

# TOPOLOGICALLY SLICE KNOTS OF SMOOTH CONCORDANCE ORDER TWO

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**ABSTRACT.** The existence of topologically slice knots that are of infinite order in the knot concordance group followed from Freedman's work on topological surgery and Donaldson's gauge theoretic approach to 4-manifolds. Here, as an application of Ozsváth and Szabó's Heegaard-Floer theory, we show the existence of an infinite subgroup of the smooth concordance group generated by topologically slice knots of concordance order two. In addition, no non-trivial element in this subgroup can be represented by a knot with Alexander polynomial one.

## 1. INTRODUCTION.

In [7] Fox and Milnor defined the smooth knot concordance group  $\mathcal{C}$ . Their proof that  $\mathcal{C}$  is infinite quickly yields an infinite family of distinct elements of order two. Results of Murasugi [24] and Tristram [39] demonstrated that  $\mathcal{C}$  also contains a free summand of infinite rank. This work culminated in Levine's construction [19] of a surjective homomorphism  $\phi: \mathcal{C} \rightarrow \mathcal{G}$ , where  $\mathcal{G}$  is an algebraically defined group isomorphic to the infinite direct sum  $\mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$ .

Classical surgery theory allowed Levine to prove that  $\phi$  is an isomorphism in high (odd) dimensions. The first distinction between classical and high-dimensional concordance was seen in the work of Casson and Gordon [2], who showed that the kernel of  $\phi$  is nontrivial; this was followed by a proof by Jiang [18] that  $\ker(\phi)$  contains a subgroup isomorphic to  $\mathbb{Z}^\infty$ . In [20] it was shown that  $\ker(\phi)$  also contains a subgroup isomorphic to  $\mathbb{Z}_2^\infty$ .

The work of Donaldson [4] and Freedman [8, 9] on smooth and topological 4-manifolds, respectively, revealed further subtlety present in low-dimensional concordance. One can define a concordance group  $\mathcal{C}^{top}$  in the topological, locally flat, category. The distinction between the smooth and topological categories is highlighted by considering the kernel of the natural surjection  $\mathcal{C} \rightarrow \mathcal{C}^{top}$ . This kernel is generated by topologically slice knots, and we denote it  $\mathcal{C}_{TS}$ . To underscore the importance of  $\mathcal{C}_{TS}$  it should be mentioned that a single non-trivial element in  $\mathcal{C}_{TS}$  implies the existence of a smooth 4-manifold homeomorphic, but not diffeomorphic, to  $\mathbb{R}^4$ . Several people, including Akbulut and Casson, observed that the results of Donaldson and Freedman can be used to produce non-trivial elements in  $\mathcal{C}_{TS}$  (see [3]), but until recently little was known about the structure of  $\mathcal{C}_{TS}$ . Using Donaldson theoretic techniques developed by Fintushel-Stern [6] and Furuta

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[10], Endo [5] proved that  $\mathcal{C}_{TS}$  contains a subgroup isomorphic to  $\mathbb{Z}^\infty$  (see also [13, 14, 15] for other constructions of infinite rank free subgroups). More recently, techniques derived from Heegaard-Floer theory were used to show that  $\mathcal{C}_{TS}$  contains a *summand* isomorphic to  $\mathbb{Z}^3$  [21, 22, 23]. With the abundance of 2-torsion in  $\mathcal{C}$ , one might expect that  $\mathcal{C}_{TS}$  likewise has such torsion. Despite this, producing torsion classes in  $\mathcal{C}_{TS}$  is quite difficult since one needs a manifestly smooth invariant to detect them. Many of the known techniques for analyzing  $\mathcal{C}_{TS}$ , however, fail at detecting torsion classes (e.g. the Ozsváth-Szabó [31] or Rasmussen [38] concordance invariants). Our main result shows that like the concordance group,  $\mathcal{C}_{TS}$  has an abundance of 2-torsion.

**Theorem 1.**  *$\mathcal{C}_{TS}$  contains a subgroup isomorphic to  $\mathbb{Z}_2^\infty$ .*

Freedman's work [8, 9] implied that all knots of Alexander polynomial one are topologically slice, and these knots provided all the early examples of nontrivial elements in  $\mathcal{C}_{TS}$ . However, in [14] it was shown that  $\mathcal{C}_{TS}$  in fact contains a subgroup isomorphic to  $\mathbb{Z}^\infty$  with no nontrivial element represented by a knot with Alexander polynomial one. Here we extend this to 2-torsion.

**Theorem 2.** *The subgroup from Theorem 1 can be chosen so that no nontrivial member is representable by a knot with Alexander polynomial one.*

To prove these theorems we consider knots  $K_{J,n}$  as illustrated in Figure 1. The knot drawn bounds an evident genus one Seifert surface. The bands in that surface are tied in knots  $J$  and  $-J$  and have  $n$  and  $-n$  full twists, where  $n > 0$ . An important special case occurs when  $U$  is the unknot, whereby  $K_{U,1}$  is the figure eight knot. We have the following easy proposition:

**Proposition 1.1.**  *$K_{J,n}$  is negative amphicheiral ( $K_{J,n} = -K_{J,n}$ ); in particular,  $2K_{J,n} = 0 \in \mathcal{C}$ . If  $J_1$  and  $J_2$  are concordant, then  $K_{J_1,n}$  and  $K_{J_2,n}$  are concordant.*

The amphichirality of  $K_{J,n}$  can be demonstrated just as for the case  $J = U$ . Indeed, an isotopy to  $-K_{J,n}$  is obtained by pulling the bottom band through the rectangular region and then rotating the knot  $180^\circ$  about a vertical axis running down the center of the page. The second part of the lemma follows from the fact that satellite operations descend to concordance, and  $K_{J,n}$  is a two-fold satellite operation with companions  $J$  and  $-J$ . The proposition allows for the immediate construction of two torsion elements in  $\mathcal{C}_{TS}$ .

**Corollary 1.2.** *For  $U$  the unknot,  $2(K_{J,n} \# K_{U,n}) = 0 \in \mathcal{C}$ . If  $J$  is topologically slice, then  $K_{J,n} \# K_{U,n}$  is topologically slice; that is,  $K_{J,n} \# K_{U,n} \in \mathcal{C}_{TS}$ .*

Let  $D = Wh(T_{2,3}, 0)$  denote the untwisted Whitehead double of  $T_{2,3}$  and let  $D_k$  denote  $kD$ . The proofs of Theorem 1 and Theorem 2 are immediate corollaries of the following theorem.

**Theorem 3.** *There exists an infinite set of positive integers  $\mathcal{N} = \{n_i\}_{i \in \mathbb{Z}}$  and for each  $n \in \mathcal{N}$  a positive integer  $k_n$  with the following property. If  $L$  is any knot with Alexander polynomial 1 and  $\#_n a_n(K_{D_{k_n}, n} \# K_{U,n}) \# L = 0 \in \mathcal{C}$  for a finite set of  $n \in \mathcal{N}$ , then all  $a_i \equiv 0 \pmod{2}$ .*

Clearly the knots  $K_{D_{k_n}, n} \# K_{U,n}$  generate a 2-torsion subgroup  $\mathcal{H}$  of  $\mathcal{C}_{TS}$ . Letting  $L$  be the unknot, we see that the  $K_{D_{k_n}, n} \# K_{U,n}$  are linearly independent over  $\mathbb{Z}_2$ .

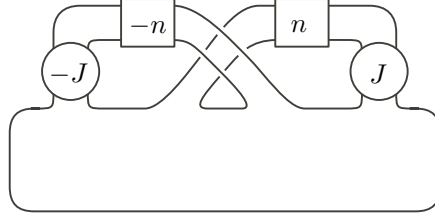


FIGURE 1.

This completes the proof of Theorem 1. Letting  $L$  be an arbitrary polynomial one knot gives the proof of Theorem 2. The proof of Theorem 3 is a consequence of Theorem 5.4 and Theorem 6.7.

## 2. PROOF OUTLINE

**2.1. Algebraic slicing obstructions.** The proof of our main results are based on considering 2-fold branched covers of  $S^3$  over  $K_{J,n}$ , which we denote  $M(K_{J,n})$ . According to [1],  $M(K_{J,n})$  has surgery description as illustrated in Figure 2, in which the meridian  $\mu$  is labeled for later reference. (In general, if a link is formed from the Hopf link by tying a local knot  $K_1$  in one component,  $K_2$  in the second, and then performing  $n_1$  and  $n_2$  surgery on the link, we denote the resulting manifold  $S^3_{n_1, n_2}(K_1, K_2)$ .) If  $J$  is reversible then  $M(K_{J,n})$  has the surgery description  $S^3_{-2n, 2n}(-2J, 2J)$ .

From this surgery description a quick calculation yields:

**Lemma 2.1.**  $H^2(M(K_{J,n})) \cong H_1(M(K_{J,n})) \cong \mathbb{Z}_{4n^2+1}$ .

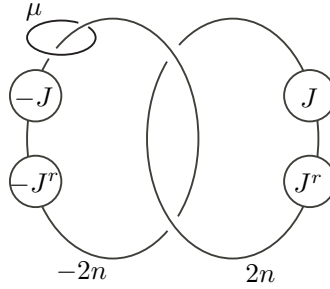


FIGURE 2.

As a special case, we note that  $M(K_{U,n})$  is given by  $(4n^2 + 1)/2n$ -surgery on the unknot:  $M(K_{U,n}) = L(4n^2 + 1, 2n)$ .

If a knot  $K$  is slice with slice disk  $D$ , then  $M(K)$  bounds the 2-fold branched cover of  $B^4$  branched over the slice disk,  $W(D)$ . In this case we have the following from [2].

**Proposition 2.2.** *The homology groups  $H_i(W(D), \mathbb{Z}_2) = 0$  for  $i \geq 1$ . The image  $I$  of the restriction map  $H^2(W(D)) \rightarrow H^2(M(K))$  is a subgroup of order satisfying  $|I|^2 = |H_1(M(K))|$ . Furthermore,  $I$  is self-annihilating with respect to the linking*

form. (Via duality, we view the linking form, usually defined on  $H_1(M)$ , as a form on  $H^2(M)$ .)

**2.2. Slicing obstructions from Heegaard-Floer theory.** Heegaard-Floer theory associates a (filtered homotopy class of) chain complex  $CF^\infty(M, \mathfrak{s})$  to a 3-manifold  $M$  with  $\text{Spin}^c$  structure  $\mathfrak{s}$ . For a manifold  $X$ , the set of  $\text{Spin}^c$  structures  $\text{Spin}^c(X)$  is in bijection with elements in  $H^2(X)$ , though not canonically so. Associated to each  $\mathfrak{s} \in \text{Spin}^c(X)$ , however, there is a first Chern class,  $c_1(\mathfrak{s}) \in H^2(X)$ , and in the case that  $H^2(X, \mathbb{Z}_2) = 0$ , the map:

$$c_1 : \text{Spin}^c(X) \rightarrow H^2(X)$$

provides a bijection which is natural with respect to the transitive action of  $H^2(X)$  on both sides and with respect to pull-back; that is

- (1)  $c_1(\mathfrak{s} + \alpha) = c_1(\mathfrak{s}) + 2\alpha$  for all  $\alpha \in H^2(X)$ , and
- (2)  $c_1(i^*\mathfrak{s}) = i^*c_1(\mathfrak{s})$ , for any map  $i : Y \rightarrow X$ . In particular, for the inclusion of a submanifold  $Y \subset X$ , we have  $c_1(\mathfrak{s}|_Y) = c_1(\mathfrak{s})|_Y$ .

Since all our manifolds satisfy  $H^2(X, \mathbb{Z}_2) = 0$ , we often denote  $\text{Spin}^c$  structures by  $\mathfrak{s}_\alpha$  for  $\alpha \in H^2(X)$ . There is an operation of conjugation of  $\text{Spin}^c$  structures taking a structure  $\mathfrak{s}$  to  $\bar{\mathfrak{s}}$ , and  $\bar{\bar{\mathfrak{s}}} = \mathfrak{s}$ .

As described in greater detail in Section 3, there is an invariant  $d(M, \mathfrak{s})$ , called the *correction term*, defined in terms of the filtered homotopy type of  $CF^\infty(M, \mathfrak{s})$ . It satisfies the following properties.

- (1)  $d(-M, \mathfrak{s}) = -d(M, \mathfrak{s})$ .
- (2)  $d(M_1 \# M_2, \mathfrak{s}_1 \# \mathfrak{s}_2) = d(M_1, \mathfrak{s}_1) + d(M_2, \mathfrak{s}_2)$ .
- (3)  $d(M, \bar{\mathfrak{s}}) = d(M, \mathfrak{s})$ .

The following theorem provides the obstruction we will use to show that knots are not smoothly slice. (The use of  $d$  as a slicing obstruction first appeared in [23], where it was applied only for the  $\text{Spin}$  structure. In [11, 17] it was used in conjunction with a careful analysis of  $\text{Spin}^c$  structures to study concordance.)

**Proposition 2.3.** *Suppose  $(W, \mathfrak{t})$  is a  $\text{Spin}^c$  4-manifold satisfying  $H_i(W, \mathbb{Q}) = 0, i > 0$ , and  $M = \partial W$ . Then  $d(M, \mathfrak{t}|_M) = 0$ .*

The proof of the main theorem is thus reduced to choosing the set of  $n$  and  $D_{k_n}$  in such a way that the estimation of values of  $d(M(K_{D_{k_n}, n}))$  is possible and sufficient for Proposition 2.3 to provide an obstruction to the triviality of the knots described in Theorem 3.

The computation of the  $d$ -invariants of  $M(K_{D_{k_n}, n})$  proceeds in the following steps.

- (1)  $\widehat{HFK}(S^3, D)$  is given by [12].
- (2) In Appendix A we describe how this determines  $CFK^\infty(S^3, D)$ . Forming the connected sum of knots corresponds to tensor products on  $CFK^\infty$ , and thus we determine  $CFK^\infty(S^3, -2D_{k_n})$ .
- (3) Letting  $Y = S^3_{-2n}(-2D_{k_n})$ , techniques from [12] (which is principally focused on  $\widehat{HFK}$ ) provides a description of  $CFK^\infty(Y, \mu, \mathfrak{s})$  in terms of the complex  $CFK^\infty(S^3, -2D_{k_n})$ , where  $\mu$  is the meridian shown in Figure 2.

- (4) We form the connected sum of  $\mu$  with  $2D_{k_n}$ , which via tensor products determines  $CFK^\infty(Y, \mu \# 2D_{k_n}, \mathfrak{s})$  for appropriate  $\text{Spin}^c$  structures.
- (5) From these homology groups we can apply results from [36] to compute  $HF^\infty(M(K_{D_{k_n}, n}), \mathfrak{s})$  for appropriate  $\mathfrak{s}$ . Background is provided in Section 4.

### 3. HEEGAARD-FLOER COMPLEXES

**3.1. Three-manifold complexes.** We let  $\mathbb{F}$  denote the field with two elements. As mentioned earlier, given a 3-manifold  $M$  with  $\text{Spin}^c$  structure  $\mathfrak{s}$ , there is an associated  $\mathbb{Z}$ -filtered graded complex  $CF^\infty(M, \mathfrak{s})$ . This complex is a free, finitely generated  $\mathbb{F}[U, U^{-1}]$ -module. The action of  $U$  lowers filtration level by one and lowers grading by two. The complex is well-defined up to filtered chain homotopy equivalence.

Added notation permits the simple representation of subcomplexes; for instance, we denote the subcomplex consisting of elements of filtration level at most  $n$  by  $CF^\infty(M, \mathfrak{s})_{\{i \leq n\}}$ . With this we can define two associated complexes,

$$CF^+(M, \mathfrak{s}) = CF^\infty(M, \mathfrak{s}) / CF^\infty(M, \mathfrak{s})_{\{i < 0\}}$$

and

$$\widehat{CF}(M, \mathfrak{s}) = CF^\infty(M, \mathfrak{s})_{\{i \leq 0\}} / CF^\infty(M, \mathfrak{s})_{\{i < 0\}}.$$

There are corresponding homology groups,  $HF^+(M, \mathfrak{s})$  and  $\widehat{HF}(M, \mathfrak{s})$ .

There will also be situations in which we must shift the gradings of elements in these chain complexes. For instance, we will write  $CF^+(M, \mathfrak{s})[k]$  for the same complex as  $CF^+(M, \mathfrak{s})$ , except with the homological grading of any element increased by  $k$ ; that is,

$$CF_*^+(M, \mathfrak{s})[k] = CF_{*-k}^+(M, \mathfrak{s}),$$

for all  $*$ .

**Definition 3.1.** The  $d$ -invariant  $d(M, \mathfrak{s})$  is given by

$$\min\{gr(\alpha) \mid \alpha \neq 0 \in HF^+(M, \mathfrak{s}) \text{ and } \alpha \in \text{Image } U^n \text{ for all } n > 0\},$$

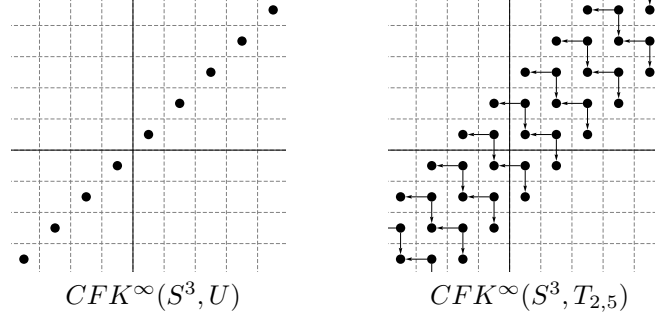
where  $gr(\alpha)$  is the homological grading.

**3.2. Knot complexes.** A knot  $K \subset M$  induces a second  $\mathbb{Z}$ -filtration of  $CF^\infty(M, \mathfrak{s})$ , which thus becomes a graded,  $\mathbb{Z} \oplus \mathbb{Z}$ -filtered complex. The  $U$  action respects the second filtration, lowering this filtration by one also. This doubly filtered complex is denoted  $CFK^\infty(M, K, \mathfrak{s})$ , and again there are associated subcomplexes such as  $CFK^\infty(M, K, \mathfrak{s})_{\{i \leq m, j \leq n\}}$ . As in the 3-manifold case, there are quotient complexes  $CFK^+(M, K, \mathfrak{s})$  and  $\widehat{CFK}(M, K, \mathfrak{s})$ . Note that ignoring the  $j$  filtration yields the corresponding complexes for  $(M, \mathfrak{s})$ .

Figure 3 illustrates the complexes for the unknot and the  $(2, 5)$ -torus knot in  $S^3$ . (For alternating knots  $K$ ,  $CFK^\infty(S^3, K)$  is determined simply from the Alexander polynomial [27].) The dots represent elements in a filtered basis, line segments indicate boundary maps. Sometimes we will not need to include arrows on the segments; the fact that the boundary map cannot increase either filtration and  $\partial^2 = 0$  will make the direction unambiguous in most the examples we consider. The gradings are not indicated in the diagram; the coordinates in the diagram correspond to the filtration, as follows: the vertical and horizontal axes in bold

separate elements of filtration levels  $-1$  and  $0$ . That is, the dot just above and to the right of the origin has filtration level  $(0, 0)$ . The action of  $U$  shifts the diagram down and to the left by one.

**Convention.** In all the cases we consider,  $CFK^\infty(M, K, \mathfrak{s})$  is filtered chain homotopy equivalent to  $C \otimes_{\mathbb{F}} \mathbb{F}[U, U^{-1}]$  for some finite  $\mathbb{Z} \oplus \mathbb{Z}$ -filtered  $\mathbb{F}$ -complex  $C$ . We will simplify our diagrams and illustrate only  $C$ , leaving out all of its  $U$  translates.



Before discussing its proof, we illustrate the theorem in Figure 4, which shows the complexes  $CFK^\infty(S^3_{-N}(K), \mu, \mathfrak{s}_m)$  for  $K = U$  and  $K = -T_{2,5}$ , with  $-3 \leq m \leq 4$ . We show only the  $\mathbb{F}$ -subcomplex that generates the full complex over  $\mathbb{F}[U, U^{-1}]$ .

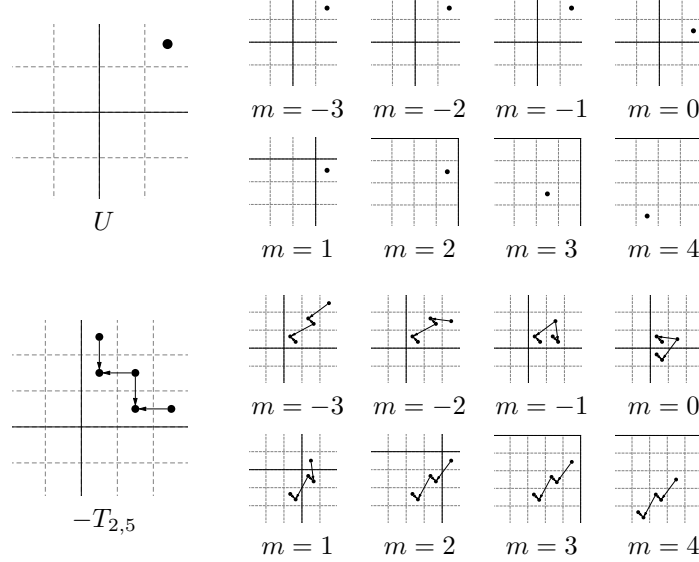


FIGURE 4.

**Proof.** The theorem refines [12, Theorem 4.1] in two directions:

- (1) [12, Theorem 4.1] determines the  $\mathbb{Z}$ -filtered chain homotopy type of  $\widehat{CFK}(S^3_{-N}(K), \mu, \mathfrak{s}_m)$ . Here we seek to understand the  $\mathbb{Z} \oplus \mathbb{Z}$ -filtered chain homotopy type of  $CFK^\infty(Y^3_{-N}(K), \mu, \mathfrak{s}_m)$ .
- (2) [12, Theorem 4.1] applies for  $N \gg 0$ . We wish to show that  $N = 2g(K)$  suffices.

The first refinement is an immediate extension of the proof from [12], so we do not belabor the details here. To begin, we note that the difference between  $S^3$  and a general 3-manifold is merely notational. The key idea from [12] was to observe that with the addition of another basepoint, the natural Heegaard diagram for  $-N$ -framed surgery on  $K$  could be made to represent the knot  $\mu \subset Y_{-N}(K)$ . The proof of [30, Theorem 4.1] shows that the  $\mathbb{Z}$ -filtered chain homotopy type of  $CF^\infty(Y_{-N}(K), \mathfrak{s}_m)$  is determined by that of  $CFK^\infty(Y, K)$ . This implies that the chain homotopy type of the complexes  $CF^-(Y_{-N}(K), \mathfrak{s}_m)$ ,  $CF^+(Y_{-N}(K), \mathfrak{s}_m)$ , and  $\widehat{CF}(Y_{-N}(K), \mathfrak{s}_m)$  are also determined by  $CFK^\infty(Y, K)$ , as they are sub, quotient, and subquotient complexes of the filtration, respectively. Now the meridian  $\mu \subset Y_{-N}(K)$  induces an additional  $\mathbb{Z}$ -filtration of any of these complexes, and [12, Theorem 4.1] determined that in the case of  $\widehat{CF}(Y_{-N}(K), \mathfrak{s}_m)$ , the  $\mathbb{Z}$ -filtration consists of two steps:

$$0 \subseteq C\{i = 0, j \geq m\} \subseteq C\{\min(i, j - m) = 0\},$$

where the subquotient on the right was identified with  $\widehat{CF}(Y_{-N}(K), \mathfrak{s}_m)$  by [30, Theorem 4.1]. In the present context, the extension to  $CFK^\infty$  follows immediately from the same strategy. To be more precise, [30, Theorem 4.1] identifies  $CF^\infty(Y_{-N}(K), \mathfrak{s}_m)$  with  $CFK^\infty(Y, K)$  via a chain map which was denoted  $\Phi$ . This isomorphism of chain complexes respects the  $\mathbb{F}[U, U^{-1}]$ -module structure of both complexes, and hence one of the  $\mathbb{Z}$ -filtrations. The additional  $\mathbb{Z}$ -filtration on  $CF^\infty(Y_{-N}(K), \mathfrak{s}_m)$  induced by  $\mu$  can be determined in exactly the same manner as it was determined for the case of  $\widehat{CF}(Y_{-N}(K), \mathfrak{s}_m)$  in [12], yielding the statement of the theorem. In both cases, the key lemma is [12, Lemma 4.2], which identifies the  $\mathbb{Z}$ -filtration induced on any given  $C\{i = \text{constant}\}$  slice in  $CF^\infty(Y_{-N}(K), \mathfrak{s}_m)$  with a two step filtration as above.

For the second refinement, recall that the proof of [12, Theorem 4.1] relied on making the surgery parameter large enough so that an entire  $\text{Spin}^c$  equivalence class of intersection points for  $Y_{-N}(K)$  was supported in the *winding region* (the region shown in [12, Figure 13]). This is achieved by a pigeonhole argument: there are only finitely many  $\text{Spin}^c$  equivalence classes that can be represented by the finitely many exterior intersection points (those points not supported in the winding region), and increasing  $N$  increases the number of  $\text{Spin}^c$  structures without bound. Once we have an entire  $\text{Spin}^c$  equivalence class supported in the winding region, we can appeal to the technique of “moving the basepoint.” In the present context this means moving the placement of the meridian and nearby collection of basepoints throughout the winding region, see [12, Theorem 4.3]. This technique allows us to use the single  $\text{Spin}^c$  equivalence class of intersection points which is supported in the winding region to represent all  $|H_1(Y)| \cdot N$  different  $\text{Spin}^c$  structures on  $Y_{-N}(K)$  (for a manifold with  $b_1(Y) > 0$ ,  $|H_1(Y)|$  should be replaced by the number of  $\text{Spin}^c$  structures on  $Y$  represented by the diagram). Thus the question is reduced to finding a topological interpretation of the number of  $\text{Spin}^c$  classes represented by exterior intersection points.

To achieve this we use a particular Heegaard diagram which is adapted to a Seifert surface for  $K$  with genus  $g$ . A similar Heegaard diagram appears in the proof of the adjunction inequality [30, Theorem 5.1], and is constructed explicitly in [29, Lemma 7.3] and [25, Proof of Theorem 2.1]. The diagram consists of a quadruple,

$$(\Sigma_k, \vec{\alpha} = \{\alpha_1, \dots, \alpha_k\}, \vec{\beta} = \{\beta_1, \dots, \beta_{k-1}, \mu, \lambda\}, \{w \cup z\}),$$

where  $(\Sigma, \vec{\alpha}, \vec{\beta} \setminus \mu)$  and  $(\Sigma, \vec{\alpha}, \vec{\beta} \setminus \lambda)$  are Heegaard diagrams for  $Y_0(K)$  and  $Y$ , respectively, and  $\{w \cup z\}$  specifies  $K$  on the latter diagram. The key features of the diagram are that

- There is a domain  $\mathcal{P}$  with  $\partial\mathcal{P} = \alpha_k \cup \lambda$  such that  $\mathcal{P} \cup \{\text{Disk bounded by } \alpha_k\}$  is isotopic to the chosen Seifert surface.
- The only  $\alpha$  curves which intersect  $\mathcal{P}$  are  $\alpha_k$  and  $\alpha_1, \dots, \alpha_{2g}$ , where  $g$ , as above, is the genus of  $K$  (= genus of  $\mathcal{P}$ ).

Now we observe that the diagram:

$$(\Sigma_k, \vec{\alpha} = \{\alpha_1, \dots, \alpha_k\}, \vec{\beta} = \{\beta_1, \dots, \beta_{k-1}, \lambda^{-N}\}, \{w\}),$$

specifies  $Y_{-N}(K)$ , where  $\lambda^{-N}$  is a simple closed curve isotopic to the resolution of  $N$  parallel copies of  $\mu^r$  and one copy of  $\lambda$ . Furthermore, with an additional point  $z'$  as in [12, Figure 15] the diagram specifies the knot  $\mu \subset Y_{-N}(K)$ . Now the generators of



$CFK^\infty(Y_N(K), \mu)$  are split into two groups, those in the *winding region* and those which are *exterior*. These latter points use  $\lambda^{-N}$  outside the winding region and are in bijection with generators for the Heegaard diagram of  $Y_0(K)$  (the diagram with  $\lambda$  as the last curve). Our bound of  $2g(K)$  in the theorem will be attained if we can argue that the total number of  $\text{Spin}^c$  equivalence classes represented by the exterior points is less than  $|H_1(Y)| \cdot 2g$ . This follows immediately from the key properties of our Heegaard diagram. Indeed, recall the first Chern class formula [29, Proposition 7.5]:

$$(3.1) \quad \langle c_1(\mathfrak{s}_w(\mathbf{x})), [\mathcal{P}] \rangle = \chi(\mathcal{P}) + 2 \sum_{x_i \in \mathbf{x}} n_{x_i}(\mathcal{P}).$$

Here,  $\mathbf{x}$  is a  $k$ -tuple generating a Heegaard Floer complex,  $[\mathcal{P}] \in H_2$  is the second homology class obtained by capping off the boundary components of a periodic domain,  $\chi(\mathcal{P})$  is the Euler measure of  $\mathcal{P}$  (which agrees with the Euler characteristic for periodic domains with all multiplicities zero or one) and  $n_{x_i}(\mathcal{P})$  is the average of the local multiplicities of  $\mathcal{P}$  in the four regions surrounding an intersection point  $x_i$ . For our particular Heegaard diagram for  $Y_0(K)$ , the right-hand side of 3.1 becomes:

$$-2g + 2\#\{x_i \in \text{interior}(\mathcal{P})\} + 2,$$

where  $-2g$  is the Euler characteristic of  $\mathcal{P}$ . The additional  $+2$  term comes from the fact that  $\alpha_k$  and  $\lambda$  do not intersect and must each contain an  $x_i \in \mathbf{x}$ . Since  $\alpha_k$  and  $\lambda$  are on the boundary of  $\mathcal{P}$  each of these two  $x_i$  have  $n_{x_i}(\mathcal{P}) = 1/2$ . Finally, the fact that there are only  $2g$  other  $\alpha$  curves which intersect  $\mathcal{P}$  implies that

$$0 \leq 2\#\{x_i \in \text{interior}(\mathcal{P})\} \leq 4g,$$

thus showing that

$$-2g + 2 \leq \langle c_1(\mathfrak{s}_w(\mathbf{x})), [\mathcal{P}] \rangle \leq 2g + 2.$$

Now the fact that  $\langle c_1(\mathfrak{s}_w(\mathbf{x})), [\mathcal{P}] \rangle$  is an even integer which vanishes if  $c_1(\mathfrak{s}_w(\mathbf{x}))$  is torsion implies that there are at most  $|H_1(Y)| \cdot (2g + 1)$  distinct  $\text{Spin}^c$  equivalence classes represented on the Heegaard diagram for  $Y_0(K)$ , and hence the same bound for the number of exterior intersection points. This completes the proof.  $\square$

#### 4. FROM KNOT COMPLEXES TO THE $d$ -INVARIANT

In general, the computation of the  $d$ -invariant of surgery on a knot  $K \subset Y$  from  $CFK^\infty(Y, K, \mathfrak{s})$  can be rather challenging; identifying patterns among the values that arise for various values of  $\mathfrak{s}$  is even more subtle. If the surgery coefficient is appropriately large, however, there are significant simplifications. This section describes the general theory and demonstrates that in our setting the simplifications that arise from the large surgery assumption do apply.

To be more specific [35, Theorem 4.1] showed that the complex  $CFK^\infty(S^3, K)$  determines  $CF^+(S_N^3(K), \mathfrak{s}_m)$  for  $N \geq 2g(K) - 1$ . In [36] this was generalized to knots in rational homology spheres  $Y$ , in which case  $CF^+(Y_N(K), \mathfrak{s}_m)$  depends on the complexes  $CFK^\infty(Y^3, K, \mathfrak{s}'_{m'})$  for specified classes  $\mathfrak{s}'_{m'}$ . However, the generalization of [36] did not specify how large the framing parameter had to be in order to apply the result. Rather, it simply showed that for sufficiently large framings such a formula exists, and then a more general formula was proved which holds for arbitrary framings in terms of a mapping cone complex. In our situation we will apply a special case of the results of [36], taking advantage of the fact that  $Y = S^3_{-2n}(-2D_{k_n})$ , and that we are performing  $2n$ -surgery on a knot formed as

the connected sum of a knot in  $S^3$  with the meridian of  $-2D_{k_n}$ . While we utilize the full mapping cone complex, our surgery parameters are chosen so that they will be large enough for the simpler formula to hold. This will manifest itself in a collapse of the mapping cone complex to a single term. In general, “large” should be taken to mean: “large in comparison to the Thurston norm of the complement”.

Here is the statement of the result we need.

**Theorem 4.1.** *Let  $K_2 \subset Y = S^3_{-N}(K_1)$  be a knot of the form  $\mu \# K'_2$  where  $\mu$  is the meridian of  $K_1$  and  $K'_2$  is a knot in  $S^3$ ; assume  $N$  is even and  $N \geq 2g(K'_2)$ . There is an enumeration of  $\text{Spin}^c$  structures on  $Y_N(K_2)$ ,  $\{\mathfrak{s}_m\}_{-N^2/2 \leq m \leq N^2/2}$ , such that  $CF^+(Y_N(K_2), \mathfrak{s}_m)$  is isomorphic to*

$$Q = CFK^\infty(S^3_{-N}(K_1), K_2, \mathfrak{s}'_{m'}) / CFK^\infty(S^3_{-N}(K_1), K_2, \mathfrak{s}'_{m'})_{\{i < 0, j < m\}}[\epsilon].$$

*The elements in the quotient  $Q$  with  $i = 0, j \leq m$  and  $i \leq 0, j = m$  are at filtration level 0 in  $CF^+(Y_N(K_2), \mathfrak{s}_m)$  and  $U$  lowers filtration level by 1; the quotient of these elements at grading 0 represents  $\widehat{CF}(Y_N(K_2), \mathfrak{s}_m)$ . The grading shift,  $\epsilon$ , is a function of  $m$  and  $N$ , and in particular, the grading shift does not depend on  $K$ , but only on its homology class.*

The rest of this section is devoted to proving Theorem 4.1.

**4.1. Heegaard diagrams,  $\text{Spin}^c$  structures, homology and surgery.** Our computation of  $HF^+(M, \mathfrak{s})$  relies on results of [36], in which the general problem of computing the Heegaard Floer homology of rational surgery on a knot in a rational homology sphere is studied. Although the manifolds we consider are in some respects fairly simple, in order to apply [36] it is essential to review some of the foundations.

The manifold  $M$  we are considering is formed by surgery on a link  $(K', K) \subset S^3$  constructed from the Hopf link by placing local knots in each component. More specifically,  $M$  is given by  $-N$  surgery on  $K'$  followed by  $N$  surgery on  $K$ . Thus, our approach to computing the Heegaard-Floer homology of  $M$  is to view it as formed by performing  $N$ -surgery on knot  $K$ , viewed as a knot in  $Y = S^3_{-N}(K')$ . We begin by considering surgery on the Hopf link, in which case  $Y = S^3_{-N}(U) = -L(N, 1)$  and  $M = L(N^2 + 1, N)$ . We then move to the more general case, encompassing the situation in which the components are knotted.

**4.2. Lens Space Heegaard Diagram.** As a starting point, we consider lens spaces  $-L(N, 1)$ . On the left in Figure 5 is a doubly pointed Heegaard diagram for  $Y = -L(2, 1)$ , which we use to illustrate the general construction.

In the lens space, the surface  $\Sigma = T^2$  represented by this diagram bounds solid tori  $U_\alpha$  and  $U_\beta$  in which the curves  $\alpha$  and  $\beta$  bound embedded disks, respectively. If we let  $\eta_\alpha$  be an arc  $w$  from  $z$  on  $\Sigma$  missing  $\alpha$  that is pushed into  $U_\alpha$  (except at its endpoints) and let  $\eta_\beta$  be an arc from  $z$  to  $w$  on  $\Sigma$  missing  $\beta$  pushed into  $U_\beta$ , the union of  $\eta_\alpha$  and  $\eta_\beta$  forms an oriented knot  $K$  in  $Y$ . Notice that once isotoped into  $U_\alpha$ ,  $K$  represents the core of  $U_\alpha$ .

The meridian to  $K$  we denote  $\mu$ . The complement of  $K$  in  $U_\alpha$  is homeomorphic to  $T^2 \times I$  with  $H_1(U_\alpha - K)$  generated by  $\mu$  and the curve  $m$  illustrated on the right in Figure 5. Notice that  $H_1(Y - K)$  is generated by  $\mu$  and  $m$ , subject to the relations  $Nm - \mu = 0$ . This is shown on the left in Figure 6, which illustrates the solid torus  $U_\alpha$ . Note that in the figure  $K$  has not yet been isotoped into  $U_\alpha$ .

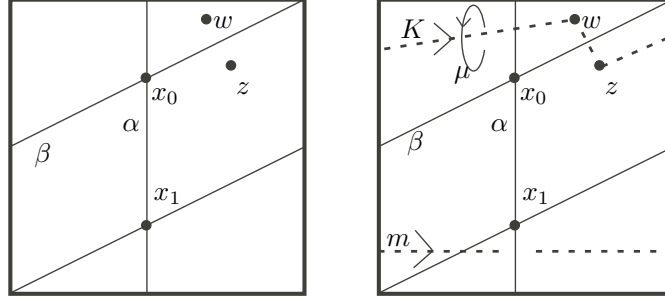
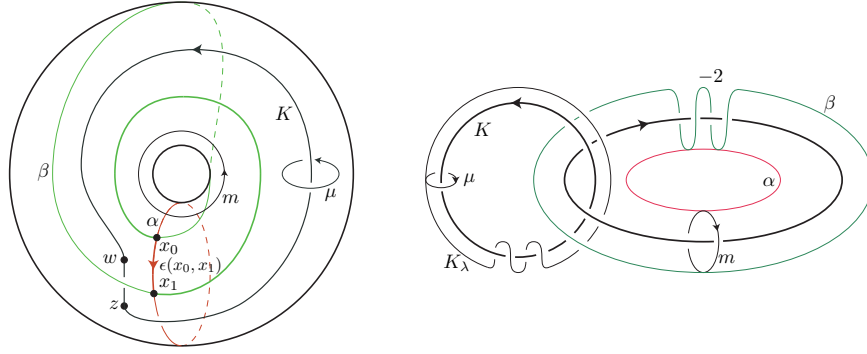


FIGURE 5. Doubly pointed Heegaard diagram


 FIGURE 6. Surgery diagram of lens space  $L(5, 2)$ 

**4.3. Relative  $\text{Spin}^c$  structures.** Associated to each intersection point,  $x_0$  or  $x_1$  in the figures and  $\{x_0, x_1, \dots, x_{N-1}\}$  for general  $-L(N, 1)$ , there is a relative  $\text{Spin}^c$  structure  $\mathfrak{s}_{w,z}(x_i) \in \text{Spin}^c(Y, K)$ . The differences between these satisfies

$$\mathfrak{s}_{w,z}(x_{i+1}) - \mathfrak{s}_{w,z}(x_i) = \text{PD}[\epsilon(x_i, x_{i+1})] \in H^2(Y \setminus \nu K, \partial)$$

where

$$\epsilon(x_i, x_{i+1}) \in \frac{H_1(\Sigma \setminus \{z, w\})}{\text{Span } \vec{\alpha} + \text{Span } \vec{\beta}} \cong H_1(Y \setminus \nu K)$$

is the class represented by a path that travels from  $x_i$  to  $x_{i+1}$  along  $\alpha$  and then from  $x_{i+1}$  to  $x_i$  along  $\beta$ . As seen from the figure, this curve is isotopic to  $m$  in  $Y \setminus \nu K$ . (In all these equations,  $i \in \mathbb{Z}/N\mathbb{Z}$ .)

There is a natural map  $G_{Y,K}: \text{Spin}^c(Y, K) \rightarrow \text{Spin}^c(Y)$  which satisfies

$$G_{Y,K}(\xi + k) - G_{Y,K}(\xi) = \iota(k),$$

where  $k \in H^2(Y, K)$ . If  $K^r$  denotes the orientation reverse of  $K$ , then

$$G_{Y,K}(\xi) - G_{Y,K^r}(\xi) = -\text{PD}[K].$$

**Comment.** As described by Turaev [40],  $\text{Spin}^c$  structures on a closed manifold correspond to equivalence classes of nonvanishing vector fields, where two are equivalent if homotopic off a ball. In the case that  $K \subset Y$  is an oriented knot, a relative  $\text{Spin}^c$  structure corresponds to a nonvanishing vector field on  $Y \setminus \nu K$  which points

outwards on the boundary. The map  $G$  is given in terms of a canonical extension of a vector field from  $Y \setminus \nu K$  to  $Y$ . See [36] for a further discussion.

**4.4.  $Y_N(K)$ .** We are interested in performing  $N$  surgery on  $K$ . To be clear about framings, in Figure 6 a push-off of  $K$ ,  $K_\lambda$ , is illustrated. The surgered manifold,  $Y_N(K)$  is built by removing a neighborhood of  $K$  and replacing it with a solid torus so that  $K_\lambda$  bounds a meridional disk in that solid torus. Note that  $H_1(Y_N(K))$  is generated by  $\mu$  and  $m$  subject to the relations  $Nm - \mu = 0$  and  $m + N\mu = 0$ . For instance in the illustrated case, with  $N = 2$ , we get  $H_1(Y_N(K)) = \mathbb{Z}/5\mathbb{Z}$ . (In fact,  $Y_2(K) = L(5, 2)$ .) In general, for  $N$  surgery on  $K$  in  $-L(N, 1)$  we end up with  $L(N^2 + 1, N) = -L(N^2 + 1, N)$ .

**4.5. The structure of  $CFK^\infty(Y, K)$ .** We continue in the case of  $Y$  a lens space, as above. Given a relative  $\text{Spin}^c$  structure  $\xi \in \text{Spin}^c(Y, K)$ , the doubly filtered chain complex  $CFK^\infty(Y, K, \xi)$  is generated by triples  $[\mathbf{x}, i, j]$  satisfying

$$\mathfrak{s}_{w,z}(\mathbf{x}) + (i - j) \cdot \text{PD}[\mu] = \xi.$$

Here,  $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$  is an intersection of the Lagrangian tori in the symmetric product of a Heegaard diagram  $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$ , and  $i, j \in \mathbb{Z}$ . For instance, for the current case, we have illustrated examples in Figure 7, where  $x$  can denote any of the  $x_i$ . The value of  $\xi$  is written beneath each of the  $\mathcal{T}(\xi)$ . (The shading in these diagrams becomes relevant later.)

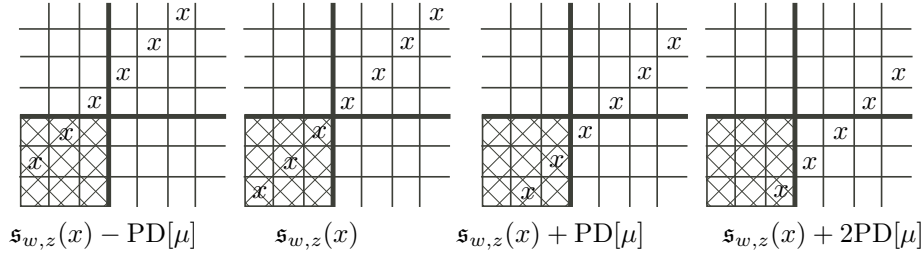


FIGURE 7.  $CFK^\infty(Y, K)$

Every relative  $\text{Spin}^c$  structure is of the form  $\mathfrak{s}_{w,z}(x_0) + k\text{PD}[\mu]$  for some  $k \in \mathbb{Z}$ , and this provides a correspondence between  $\text{Spin}^c(Y, K)$  and  $\mathbb{Z}$ . Since  $\mu = Nm$ , the set of relative  $\text{Spin}^c$  structures associated to each  $x_i$  is a coset of  $N\mathbb{Z} \subset \mathbb{Z}$ . Also, since  $\mathfrak{s}_{w,z}(x_i) - \mathfrak{s}_{w,z}(x_0) = i\text{PD}[\mu]$ , for different  $x_i$  the sets are distinct cosets.

**4.6. Enumerating Relative  $\text{Spin}^c$  Structures for the manifolds at hand.** We now move to our particular setting, when  $Y$  is a manifold constructed as surgery on a knot  $K'$  in  $S^3$  and  $K \subset Y$  is a knot formed from the meridian of  $K'$  by taking the connected sum with a knot in  $S^3$ .

The surjective map

$$G_{Y_N(K), K}: \text{Spin}^c(Y_N(K), K) \rightarrow \text{Spin}^c(Y_N(K))$$

is defined by taking a canonical extension of a vector field on  $Y_N(K) \setminus \nu K$  to  $Y_N(K)$  (see [36, Section 2.2]). There is also a corresponding map  $G_{Y, K}: \text{Spin}^c(Y, K) \rightarrow \text{Spin}^c(Y)$ . Note that there is a natural diffeomorphism of  $Y_N(K) \setminus \nu K \rightarrow Y \setminus \nu K$ , which provides an identification of  $\text{Spin}^c(Y_N(K), K)$  with  $\text{Spin}^c(Y, K)$ .

Since  $H^2(Y_N(K), K) \cong \mathbb{Z}$ , we can enumerate the relative  $\text{Spin}^c$  structures on  $(Y_N(K), K)$  by integers, where  $\underline{s}_i$  is the relative  $\text{Spin}^c$  structure satisfying

$$i = \frac{1}{2} \langle c_1(\underline{s}_i) - PD([\mu]), [F, \partial F] \rangle.$$

Here,  $[F, \partial F] \in H_2(Y_N(K) \setminus K, \partial) \cong \mathbb{Z}$  is a generator, and  $c_1(\underline{s}_i) \in H^2(Y_N(K) \setminus K, \partial)$  is the relative Chern class of the relative  $\text{Spin}^c$  structure, defined with respect to a trivialization of the two-plane field dual to the defining vector field along the boundary (see [34, Page 627] or [26] for more details). We have an analogous labeling for relative  $\text{Spin}^c$  structure for  $(Y, K)$  by  $\underline{t}_i$ . With the identification just mentioned,  $\underline{s}_i = \underline{t}_i$ . In general, this choice of enumeration is not essential and is used simply to provide coordinates in the discussion that follows.

Let  $\mathfrak{s}$  be some fixed  $\text{Spin}^c$  structure on  $Y_N(K)$ . Define  $S(\mathfrak{s}) = G_{Y_N(K), K}^{-1}(\mathfrak{s})$ . We have that  $S(\mathfrak{s}) = \{\underline{s}_{k+(N^2+1)j}\}$  for  $j \in \mathbb{Z}$  and some  $k$ ,  $0 \leq k \leq N^2$ . Via the identification described above, we can also write  $S(\mathfrak{s}) = \{\underline{t}_{k+(N^2+1)j}\}$  for  $j \in \mathbb{Z}$  and some  $k$ ,  $0 \leq k \leq N^2$ . For each fixed value of  $k$ , there exists an  $\mathfrak{s} \in \text{Spin}^c(Y_N(K))$  such that  $\underline{t}_k \in S(\mathfrak{s})$ ; this follows from the surjectivity of  $G_{Y_N(K), K}$ .

**4.7. The Mapping Cone.**  $HF^+(Y_N(K), \mathfrak{s})$  can be computed as the homology of a mapping cone complex built from  $CFK^\infty(Y, K)$  via a construction from Ozsváth and Szabó which we now recall. We use the notation of [35, 36], and refer the reader there for more details.

Letting  $S$  be as above, there are complexes

$$\begin{aligned} \mathbb{A}_{\mathfrak{s}}^+(Y, K) &= \oplus_{\underline{\xi} \in S} A_{\underline{\xi}}^+(Y, K), \\ \mathbb{B}_{\mathfrak{s}}^+(Y, K) &= \oplus_{\underline{\xi} \in S} B_{\underline{\xi}}^+(Y, K). \end{aligned}$$

Here

$$A_{\underline{\xi}}^+(Y, K) = CFK^\infty(Y, K, \underline{\xi})_{\{\max(i, j) \geq 0\}},$$

and

$$B_{\underline{\xi}}^+(Y, K) = CF^+(Y, G_{Y, K}(\underline{\xi})).$$

We can write

$$CF^+(Y, G_{Y, K}(\underline{\xi})) = CFK^\infty(Y, K, \underline{\xi})_{\{i \geq 0\}},$$

where in the term on the right of the equality,  $K$  has provided a filtration of  $B_{\underline{\xi}}^+(Y, K)$ .

There are maps:

$$v_{\underline{\xi}}^+ : A_{\underline{\xi}}^+(Y, K) \rightarrow B_{\underline{\xi}}^+(Y, K)$$

and

$$h_{\underline{\xi}}^+ : A_{\underline{\xi}}^+(Y, K) \rightarrow B_{\underline{\xi} + \text{PD}[K_\lambda]}^+(Y, K).$$

The map  $v$  is given by the projection map onto the quotient complex of  $A_{\underline{\xi}}^+(Y, K)$  consisting of triples  $[\mathbf{x}, i, j]$  with  $i \geq 0$ , the so-called vertical complex. The map  $h$  is more subtle. Interchanging the roles of  $i$  and  $j$  replaces  $K$  with  $K^r$ , its reverse. The associated filling map for  $K^r$  is denoted  $G_{Y, K^r}$ . Because of the string reversal,  $G_{Y, K^r}(\underline{\xi}) = G_{Y, K}(\underline{\xi}) + PD(K)$ . Thus, if we simply take the quotient corresponding to the horizontal projection, the target of this chain map is a complex homotopy equivalent to  $CF^+(Y, G_{Y, K}(\underline{\xi}) + PD(K))$ . The map  $h_{\underline{\xi}}^+$  is given by horizontal projection, followed by this chain homotopy equivalence.

We now want to consider the set  $S(\mathfrak{s})$  in terms of relative  $\text{Spin}^c$  structures on  $(Y, K)$ . To do so we write

$$S(\mathfrak{s}) = \{\mathfrak{t}_{k+(N^2+1)j}\}$$

for some fixed  $k, 0 \leq k \leq N^2$  and  $j \in \mathbb{Z}$ . This set  $S$  can be partitioned according to the  $N$  possible images  $G_{Y,K}(\mathfrak{t}_{k+(N^2+1)j}) \in \text{Spin}^c(Y)$ . If we fix  $k$ , let  $0 \leq l \leq N-1$ , and let  $\mathfrak{t}_l = G_{Y,K}(\mathfrak{t}_l)$ . Then  $S$  can be written as

$$\bigcup_{0 \leq l \leq N-1} \left( \{\mathfrak{t}_{[j(N^2+1)+(l-k)N]N+l} \}_{j \in \mathbb{Z}} \right).$$

Recalling that  $\mu = Nm$ , this can be rewritten as

$$\bigcup_{0 \leq l \leq N-1} \left( \{\mathfrak{t}_{l+[j(N^2+1)+(l-k)N]PD(\mu)} \}_{j \in \mathbb{Z}} \right).$$

Deriving the following formula is rather delicate, but its validity is easily checked:

$$l + [j(N^2+1) + (l-k)N]N = l \pmod{N}$$

and

$$l + [j(N^2+1) + (l-k)N]N = k \pmod{N^2+1}.$$

**4.8. Reduction to a finite complex.** From this discussion, it is apparent that in general the  $(\mathbb{A}, \mathbb{B})$  complex is fairly complicated. In this subsection we observe that it always reduces to a complex that is a quotient of a finite dimensional complex over  $\mathbb{F}[U, U^{-1}]$ . In the next subsection we observe that in our special case, the complex reduces to a single  $A_\xi$  term.

Consider the complexes  $\mathbb{A} = \oplus A_i$  and  $\mathbb{B} = \oplus B_i$ , joined by the chain map  $D$  as illustrated below. We denote the mapping cone complex of  $D$  by  $\mathbb{C}$ . Since  $CFK^\infty$  is finitely generated over  $\mathbb{F}[U, U^{-1}]$ , it follows that  $v: A_i \rightarrow B_i$  is an isomorphism for all large  $i$ , and  $h: A_i \rightarrow B_{i+1}$  is an isomorphism as  $i$  goes to negative infinity. The diagram below presents a special case.

$$\begin{array}{ccccccccccc} A_{-3} & A_{-2} & A_{-1} & A_0 & A_1 & A_2 & & & & & \\ \swarrow \cong & \downarrow v & \swarrow \cong & \downarrow v & \swarrow h & \downarrow v & \swarrow h & \downarrow \cong & \swarrow h & \downarrow \cong & \swarrow h \\ \dots & B_{-2} & B_{-1} & B_0 & B_1 & B_2 & B_3 & & & & \dots \end{array}$$

In this example, we have the following subcomplex.

$$\begin{array}{ccccccc} A_{-3} & A_{-2} & & A_1 & A_2 & & \\ \swarrow \cong & \downarrow & \swarrow \cong & \downarrow \cong & \swarrow & \downarrow \cong & \swarrow \\ \dots & B_{-2} & B_{-1} & B_1 & B_2 & B_3 & \dots \end{array}$$

The restriction of  $D$  to this subcomplex, which we denote  $D'$ , induces an isomorphism  $D'_*: H_*(\mathbb{A}') \rightarrow H_*(\mathbb{B}')$ . Injectivity is evident; surjectivity follows from the fact that for the right portion of the complex,  $(h \circ v^{-1})^k$  is the 0 map for large  $k$  and for the left portion of the complex,  $(v \circ h^{-1})^k$  is the 0 map. There is a long exact sequence

$$\mathbb{B}' \rightarrow \mathbb{C}' \rightarrow \mathbb{A}'$$

with connecting homomorphism given by  $D'$ . Thus,  $H_*(\mathbb{C}') = 0$ .

Consider next the exact sequence  $\mathbb{C}' \rightarrow \mathbb{C} \rightarrow \mathbb{C}/\mathbb{C}'$ ; it leads to a long exact sequence, and we see that  $H_*(\mathbb{C}, \mathbb{C}') = H_*(\mathbb{C})$ . That is, the homology of  $\mathbb{C}$  is the homology of the complex

$$\begin{array}{ccc} A_{-1} & & A_0 \\ & \searrow & \downarrow \\ & & B_0 \end{array}$$

Notice that had  $h: A_{-1} \rightarrow B_0$  also been an isomorphism in this example, then the complex would have reduced to a single term,  $A_0$ . This occurs in the cases of lens spaces that arise in our work,  $L(N^2 + 1, N)$ . We will see in the next section that this total collapse also occurs for our manifolds  $M$ .

**4.9. General Complete Collapse of the  $(\mathbb{A}, \mathbb{B})$  Complex.** In the case of lens spaces, the complexes  $A_i$  are all of the form  $(C \otimes_{\mathbb{F}} F[U, U^{-1}])\{0, k_i\}$ , where  $C$  is a 1-dimensional doubly filtered  $\mathbb{F}$  vector space at filtration level  $(0, 0)$ ,  $\langle x_i \rangle$ , with  $j$ -filtering shift  $k_i$  for appropriate integers  $k_i$ . It thus follows quickly that there is an  $a$  such that  $v_i$  is an isomorphism for all  $i \geq a$  and  $h_i$  is an isomorphism for all  $i \leq a - 1$ . This explains our comment above that for lens spaces there is a complete collapse of the  $(\mathbb{A}, \mathbb{B})$  complex to a single  $A_i$ .

In the more general situation that appears for our  $M$ , the complexes  $A_i = (C_{\bar{i}} \otimes_{\mathbb{F}} F[U, U^{-1}])\{0, k_i\}$  do not have  $C_{\bar{i}}$  1-dimensional, where  $\bar{i} = i \bmod n$  for some  $n$ , and thus the  $C^\infty$  complex is not restricted to a single diagonal. Instead, it lies in a band; in Figure 8 we illustrate a case in which the band is of height six.

Notice that in the example illustrated in Figure 8 the vertical quotient is not an isomorphism, but the horizontal quotient is. In general, one of the two maps will be an isomorphism unless the origin is contained in the band. Furthermore, if this band is shifted up (by  $-2$  or more) then  $h$  continues to be an isomorphism, and if it is shifted down by 7 or more, the vertical map becomes an isomorphism.

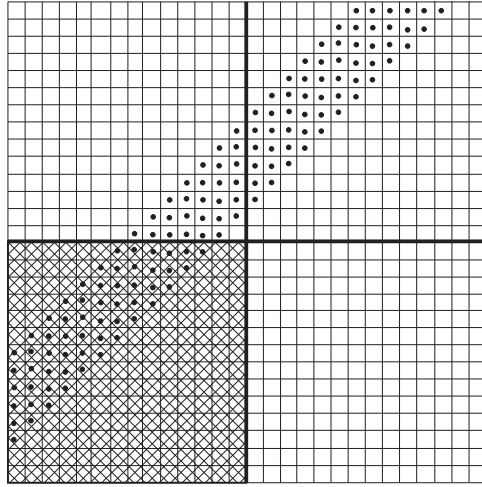


FIGURE 8.

Recall now that in our decomposition  $\mathbb{A}_{\underline{s}}^+(Y, K) = \oplus_{\xi \in S} \mathbb{A}_{\underline{\xi}}^+(Y, K)$  we have

$$S = \bigcup_{0 \leq l \leq N-1} (\{\mathfrak{t}_l + [j(N^2 + 1) + (l - k)N]PD(\mu)\}_{j \in \mathbb{Z}}).$$

In order to state the next result, let the width  $w(C)$  of a doubly filtered complex be defined as:  $w(C) = \max(i - j) - \min(i - j) + 1$ , where the minimum and maximum are taken over all pairs  $(i, j)$  such that there is a nontrivial filtered generator of filtered degree  $(i, j)$ . Roughly,  $w(C)$  represents the width of the narrowest  $U$ -invariant band which contains the full complex. The width determines the Thurston norm of the knot complement [26, 33].

**Theorem 4.2.** *Suppose that for each  $\mathfrak{t}_l \in \text{Spin}^c(Y, K)$  with  $0 \leq l \leq N - 1$ , the complex  $C = CFK^\infty(Y, K, \mathfrak{t}_l)$  satisfies  $w(C) \leq N$ . Then the complex  $\mathbb{A} \rightarrow \mathbb{B}$  that determines  $HF^+(Y_N(K), \mathfrak{s})$  collapses to a single  $A_i$  for some  $i$ .*

**Proof.** Recall that  $k$  is some fixed integer,  $0 \leq k \leq N^2$ . For simplicity, denote  $CFK^\infty(Y, K, \mathfrak{t}_l)$  by  $A'_l$  for  $0 \leq l < N$ . Then the  $A_i$  that occur are ordered as follows if we begin with  $l = 0$  and  $j = 0$ :

$$\begin{aligned} \dots A'_{N-1}\{-N-1-kN\}, A'_0\{-kN\}, A'_1\{(1-k)N\}, \dots, \\ \dots A'_{N-1}\{(N-1-k)N\}, A'_0\{N^2+1-kN\}, \dots \end{aligned}$$

Notice that the shifts increase by  $N$ , or when going from  $A'_{N-1}$  to  $A'_0$ , by  $N+1$ . It follows that at most one of  $A_i$  is in a band which includes the origin, with all greater  $A_i$  being in bands below the origin and all lesser  $A_i$  being in bands above the origin. Thus the complex collapses to a single  $A_i$ , as desired.  $\square$

For our theorem, we are performing surgery on a knot with  $CFK^\infty$  complex of the form  $CFK(S_{-N}^3(K_{-2D_{k,n}}), \mu, \mathfrak{s}) \otimes^{2k} CFK^\infty(S^3, D)$ , where  $D = Wh(T_{2,3}, 0)$  and  $\mu$  is the meridian to the mirror image of this knot. By the refiltering theorem, the complex  $CFK(S_{-N}^3(K_{-2D_{k,n}}), \mu, \mathfrak{s})$  has width at most 2. The complex  $CFK^\infty(S^3, D)$  has width 3, as shown in Appendix A. A simple exercise gives the addition formula,  $w(C_1 \otimes C_2) = w(C_1) + w(C_2) - 1$ . It follows that the complex  $C$  satisfies  $w(C) = 2k + 2$ . Combining these results with Theorem 4.2 yields:

**Corollary 4.3.** *The  $\mathbb{A}$ - $\mathbb{B}$ -complex used to compute  $CF^\infty$  for the space  $M(K_{D_{k,n}})$  collapses to a single  $A_i$  factor if  $2k + 2 \leq 2n$ . That is, if  $k \leq n - 1$ .*

## 5. EXAMPLES.

We now describe the explicit examples for which estimates of the relevant  $d$  invariants are accessible and yield the desired results. Specifically, we find the appropriate set of integers  $\mathcal{N}$  and the corresponding set  $\{k_i\}$ . The choices here might appear mysterious at first, so we begin by offering a bit of motivation.

In showing that  $(K_{D_{k_n}, m} \# K_{U, m}) \# (K_{D_{k_n}, n} \# K_{U, n})$  is not slice, we must identify the appropriate set of  $\text{Spin}^c$  structures on

$$M(K_{D_{k_m}, m}) \# M(K_{U, m}) \# M(K_{D_{k_n}, n}) \# M(K_{U, n}).$$

This set of structures corresponds to a metabolizer of the linking form on the first homology,  $(\mathbb{Z}_{4m^2+1})^2 \oplus (\mathbb{Z}_{4n^2+1})^2$ . If  $4m^2 + 1$  and  $4n^2 + 1$  are distinct primes, identifying these metabolizers would not be difficult. Unfortunately, it is not known if there are an infinite set of such primes. However, a fairly simple argument shows



that one can choose the  $n$  so that the set of values  $4n_i^2 + 1$  are relatively prime, each with few prime factors. This will be sufficient for our needs.

### 5.1. Number theoretic results.

**Theorem 5.1.** *There is an infinite set  $\mathcal{N}$  of natural numbers  $n_i$  satisfying:*

- (1) *The values  $\{4n_i^2 + 1\}$  are relatively prime, and*
- (2) *If  $4n_i^2 + 1 = \prod_{j=1}^r p_j^{a_j}$  for a distinct set of primes  $p_j$ , then  $2^r < \lfloor n/2 \rfloor$ .*

**Proof.**

**1.** Suppose that  $n_i$ ,  $i < m$ , have been chosen so that the  $\{4n_i^2 + 1\}$  are relatively prime. Consider the set of all prime divisors  $p_\alpha$  of all the  $\{4n_i^2 + 1\}_{i < m}$ . For each  $\alpha$  there is a  $c_\alpha \not\equiv 0 \pmod{p_\alpha}$  so that the equation  $4n^2 + 1 = c_\alpha$  has a solution modulo  $p_\alpha$  for  $n$ . Call that solution  $n_\alpha$ . By the Chinese Remainder Theorem, there is an  $n$  such that  $n = n_\alpha \pmod{p_\alpha}$  for all  $\alpha$ . For that  $n$ ,  $4n^2 + 1 = c_\alpha \not\equiv 0 \pmod{p_\alpha}$ , so  $4n^2 + 1$  is relatively prime to all  $p_\alpha$ , and hence also relatively prime to all  $4n_i^2 + 1$ , for all  $i$ .

**2.** For each  $N > 0$ , write  $N$  as  $N = \prod_{j=1}^r p_j^{a_j}$  with the  $p_j$  an increasing sequence of distinct primes. Fix values  $\alpha > 0$  and  $\beta > 0$ . (Our particular interest is  $\alpha = 1/2$  and  $\beta < 1/4$ .) We prove that for all large  $N$ ,  $2^r \leq \beta N^\alpha$ .

Note that  $\prod_{i=1}^r p_i^{a_i} \geq r!$ . Applying Sterling's Approximation, we have

$$N = \prod_{i=1}^r p_i^{a_i} \geq \left(\frac{r}{e}\right)^r.$$

Suppose now that  $2^r \geq \beta N^\alpha$ . Apply the logarithm to this inequality to find

$$r \geq a_1 \ln N + b_1$$

for some  $a_1 > 0$  and  $b_1$ , independent of  $N$  (and  $r$ ). Thus,

$$N \geq \left(\frac{a_1 \ln N + b_1}{e}\right)^{(a_1 \ln N + b_1)}$$

which we rewrite using new constants ( $a_2 > 0$  and  $b_2$ ) as

$$N \geq (a_2 \ln N + b_2)^{(a_1 \ln N + b_1)}.$$

Taking logarithms again, we find

$$\ln N \geq (a_1 \ln N + b_1) \ln(a_2 \ln N + b_2).$$

Clearly, for large  $N$  this inequality does not hold.  $\square$

### 5.2. Metabolizers.

**Theorem 5.2.** *If  $\mathcal{K} = K_{D_{k_n}, n} \# K_{U, n} \# L = 0 \in \mathcal{C}$ , for some knot  $L$  with  $|H_1(M(L))|$  relatively prime to  $4n^2 + 1$ , then  $d(M(K_{D_{k_n}, n}), \mathfrak{s}_{z_1}) + d(M(K_{U, n}), \mathfrak{s}_{z_2}) = 0$  for all  $(z_1, z_2)$  in some metabolizer for the linking form on  $\mathbb{Z}_{4n^2+1} \oplus \mathbb{Z}_{4n^2+1}$ .*

**Proof.** There is some metabolizer for the 2-fold cover of  $\mathcal{K}$  on for which the  $d$ -invariants vanish for all corresponding  $\text{Spin}^c$  structures. Since the order of the homology of the 2-fold cover of  $L$  is prime to that of  $K_{D_{k_n}, n} \# K_{U, n}$ , this metabolizer splits as the direct sum of metabolizers,  $\mathcal{M}_1 \oplus \mathcal{M}_2$ , for the 2-fold covers of  $K_{D_{k_n}, n} \# K_{U, n}$  and of  $L$ , respectively. Let  $(z_1, z_2) \oplus (0) \in \mathcal{M}_1 \oplus \mathcal{M}_2$ , we see that

$$d(M(K_{D_{k_n}, n}), \mathfrak{s}_{z_1}) + d(M(K_{U, n}), \mathfrak{s}_{z_2}) + d(M(L), \mathfrak{s}_0) = 0.$$

But, since  $L$  is order two (being concordant to  $K_{D_{k_n},n} \# K_{U,n}$ ),  $d(M(L), \mathfrak{s}_0) = 0$ , completing the argument.  $\square$

We have now reduced the necessary identification of  $\text{Spin}^c$  structures to finding metabolizers for the linking form on  $(\mathbb{Z}_{4n^2+1})^2$ .

**Theorem 5.3.** *For a fixed non-degenerate linking form on  $(\mathbb{Z}_{4n^2+1})$ , each metabolizer for the double of this form, on  $(\mathbb{Z}_{4n^2+1})^2$ , either contains an element  $(0, c)$  with  $c \neq 0$  or is generated by an element  $(1, b)$  where  $b$  has order  $4n^2 + 1$ .*

**Proof.** The metabolizer  $\mathcal{M}$  is generated by two elements,  $\{(a, b), (c, d)\}$ . Using Gauss-Jordan elimination, we see it is generated by a pair of elements  $\{(a, b), (0, c)\}$ . If  $c$  is nonzero, we are done. If not,  $\mathcal{M}$  is generated by a single element  $(a, b)$ . Suppose that  $a$  is divisible by some prime factor  $p$  of  $4n^2 + 1$ . The vanishing of the linking form on  $\mathcal{M}$  implies that  $a^2 + b^2$  is divisible by  $4n^2 + 1$ , and in particular by  $p$ . Since  $a$  is divisible by  $p$ ,  $b$  would also be divisible by  $p$ , but in that case we have that  $(a, b)$  would not be of order  $4n^2 + 1$ . Thus,  $a$  is relatively prime to  $4n^2 + 1$ , and after taking an appropriate multiple, we can assume that  $a = 1$ , so  $\mathcal{M}$  is generated by  $(1, b)$  for some  $b$ . But as we have just argued, this  $b$  must be relatively prime to  $4n^2 + 1$ , as desired.  $\square$

### 5.3. $d$ -invariant constraints.

**Theorem 5.4.** *With the choice of  $\mathcal{N}$  described above, if*

$$(K_{D_{k_n},n} \# K_{U,n}) \# L = 0 \in \mathcal{C},$$

*where the order of  $H_1(M(L))$  is prime to  $4n^2 + 1$ , then there is a  $b' \in \mathbb{Z}_{4n^2+1}$  satisfying  $b'^2 = 1 \pmod{4n^2+1}$  such that  $d(M(K_{D_{k_n},n}), \mathfrak{s}_x) = d(M(K_{U,n}), \mathfrak{s}_{b'x})$  for all  $x \in \mathbb{Z}_{4n^2+1}$ .*

**Proof.** We have seen that there is some number  $b$  relatively prime to  $4m^2 + 1$  such that  $(a, b) \oplus (0, 0)$  is in the metabolizer for the linking form on  $(\mathbb{Z}_{4m+1})^2 \oplus (\mathbb{Z}_{4n+1})^2$ . Thus,

$$d(M(K_{D_m,m}), \mathfrak{s}_1) + d(M(K_{U,m}), \mathfrak{s}_b) + d(M(K_{D_n,n}), \mathfrak{s}_0) + d(M(K_{U,n}), \mathfrak{s}_0) = 0.$$

Since for every choice of  $J$ , the knot  $K_{J,m}$  is of order two, we have  $2d(M(K_{J,n}), \mathfrak{s}_0) = 0$ , so  $d(M(K_{J,n}), \mathfrak{s}_0) = 0$ . Thus,

$$d(M(K_{D_m,m}), \mathfrak{s}_1) + d(M(K_{U,m}), \mathfrak{s}_b) = 0.$$

Since  $K_{U,m}$  is of order 2, there is a  $b'$  such that  $b^2 + b'^2 = 0 \pmod{4m^2+1}$  and for which

$$d(M(K_{U,m}), \mathfrak{s}_b) + d(M(K_{U,m}), \mathfrak{s}_{b'}) = 0.$$

Combining these, we have:

$$d(M(K_{D_m,m}), \mathfrak{s}_1) = d(M(K_{U,m}), \mathfrak{s}_{b'}) = 0.$$

Recall that  $b^2 = -1 \pmod{4m^2+1}$ , so  $b'^2 = 1 \pmod{4n^2+1}$ , as desired. By considering multiples of the class  $(1, b')$  the full result is achieved.  $\square$

**5.4. Counting the set of possible  $d$ -invariant constraints.** A final observation concerns the number of possible values of  $b'$  which need to be considered. That is, how many solutions are there to the equation  $b'^2 = 1 \pmod{4n^2 + 1}$ ? Observe that if  $4n^2 + 1 = \prod_{i=1,r} p_i^{a_i}$ , then any solution  $b'$  will be a solution modulo  $p_i^{a_i}$  also. But modulo  $p_i^{a_i}$  the equation  $b'^2 = 1$  has exactly two solutions. (Here we use the fact that each  $p_i$  is odd and consider the equation  $(c-1)(c+1) = 0 \pmod{p_i^{a_i}}$ .) Knowing  $b' \pmod{p_i^{a_i}}$  for all  $i$  determines  $c$  modulo  $4n^2 + 1$  uniquely, and thus, there are at most  $2^r$  solutions. For any  $n$  in  $\mathcal{N}$  of Theorem 5.1, we can ensure that the number of  $b'$  to consider is at most  $\lfloor n/2 \rfloor - 1$ .

## 6. COMPUTATIONS

**6.1. Knot complexes.** For a given  $n$  we have defined  $D_{k_n} = k_n Wh(T_{2,3}, 0)$ ; as mentioned earlier,  $n$  will be in the set  $\mathcal{N}$  and the  $k_n$  are to be determined. We continue to denote  $Wh(T_{2,3}, 0)$  by  $D$ . In this case,  $M(K_{D_{k_n}, n})$  is given as  $(-2n, 2n)$ -surgery on the link formed from the Hopf link by replacing the first component with  $-2k_n D$  and the second component by  $2k_n D$ ; see Figure 2. A key result, which we prove in Appendix A, is the following.

**Proposition 6.1.** *The chain complex  $CFK^\infty(S^3, D)$  is filtered chain homotopy equivalent to the chain complex  $CFK^\infty(S^3, T_{2,3}) \oplus A$ , where  $A$  is an acyclic complex. If  $[x, i, j]$  is a filtered generator, then  $|i - j| \leq 1$ .*

Applying the tensor product, we see that the chain complex  $CFK^\infty(S^3, 2kD)$  is chain homotopy equivalent to the direct sum of the complex for the knot  $T_{2,4k+1}$  with a complex  $A$ , where  $A$  is some acyclic complex. If  $[x, i, j]$  is a filtered generator, then  $|i - j| \leq 2k$ . The complex  $CFK^\infty(S^3, T_{2,4k+1})$  is illustrated in Figure 9. The addition of an acyclic complex will be seen not to affect the computation of the  $d$ -invariant, (in part because its only contributions to homology are not in the image of  $U^n$  for large  $n$ ). Thus, for the moment we focus on the complex  $CFK^\infty(S^3, T_{2,4k+1})$ .

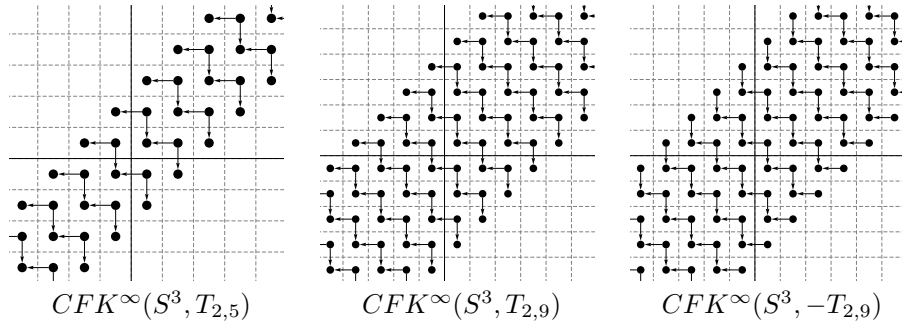


FIGURE 9.

Recalling that  $CFK^\infty(S^3, -K) = CFK^\infty(S^3, K)^*$  and using Theorem 3.2, one sees that for each  $\text{Spin}^c$  structure  $\mathfrak{s}_m$ , the complex  $CFK^\infty(S^3_{-2n}(-T_{2,4k+1}), \mu, \mathfrak{s}_m)$  is given by a complex concentrated on the diagonal and one below the diagonal. We wish to understand this complex better.

For example, Figure 10(a) illustrates the complex  $CFK(S_{-2n}^3(-T_{2,9}), \mu, \mathfrak{s}_{-3})$ , in which we have labeled two of the generators  $A$  and  $B$ . Replacing  $B$  with  $A + B$ , is a filtered change of basis, and the new complex is as shown in Figure 10(b). Notice that this has introduced an acyclic piece, which does not affect our computation of the  $d$ -invariant. Here is the general result.

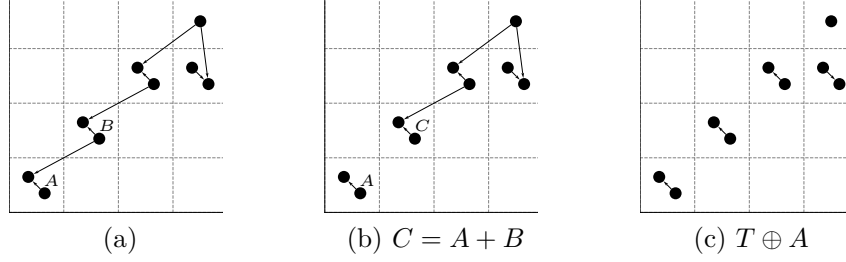


FIGURE 10.

**Theorem 6.2.** *For  $2n \geq 4k + 1$  and  $-n + 1 \leq m \leq n$ , the doubly filtered complex  $CFK^\infty(S_{-2n}^3(-2D_k), \mu, \mathfrak{s}_m)$  is chain homotopy equivalent to the complex  $C_{2n,k,m} \cong (T \oplus A) \otimes_{\mathbb{F}} \mathbb{F}[U, U^{-1}]$  where  $A$  is a finitely generated acyclic complex and  $T$  has one generator at filtration level  $(0, 0)$  or  $(0, -1)$ . More precisely, the generator of  $T$  has filtration level  $(0, 0)$  if  $m < -2k$  or  $m$  odd  $< 2k$ , and has filtration level  $(0, -1)$  if  $m \geq 2k$  or  $m$  even  $\geq -2k$ . For any filtered generator  $[x, i, j]$ ,  $|i - j| \leq 1$ .*

We next want to consider the second component of the link. Since this is obtained from the meridian of the first component by forming the connected sum with  $T_{2,4k+1}$ , and the complex for the meridian is simply that of the unknot with filtration shift, the result is as follows.

**Theorem 6.3.** *For  $2n \geq 4k + 1$  and for  $-n + 1 \leq m \leq n$ , we have*

$$CFK^\infty(S_{-2n}^3(-2D_k), 2D_k, \mathfrak{s}_m)_{i,j} = CFK^\infty(S^3, T_{2,4k+1})_{i,j-\delta} \oplus A.$$

Here,  $\delta = 0$  if  $m < -2k$  or  $m$  odd  $< 2k$ ;  $\delta = -1$  if  $m \geq 2k$  or  $m$  even  $\geq -2k$ . Cycles representing nontrivial classes of grading 0 are located at filtration levels  $i + j = 2k + \delta$ . For any filtered generator  $[x, i, j]$ ,  $|i - j| \leq 2k + 1$ .

We will need to compare this with the case of  $J$  the unknot. Here the computation is simpler.

**Theorem 6.4.** *For  $2n \geq 3$  and for  $-n + 1 \leq m \leq n$ , we have*

$$CFK^\infty(S_{-2n}^3(-U), U, \mathfrak{s}_m)_{i,j} = CFK^\infty(S^3, U)_{i,j-\delta}.$$

Here,  $\delta = 0$  if  $m \leq -1$  and  $\delta = -1$  if  $m \geq 0$ . The cycle representing a nontrivial homology class is at filtration level  $(0, \delta)$ .

**6.2. Surgery on the knot.** We first compute the difference  $d(M(K_{D_k,n}), \mathfrak{s}_n) - d(M(K_{U,n}), \mathfrak{s}_n)$  for any  $k$  with  $0 \leq k < n/2$ . Recall that  $M(K_{D_k,n})$  is also denoted by  $S_{-2n,2n}^3(-2D_k, 2D_k)$ . Noting  $n = 0 \cdot (2n) + n$ , we let  $a = 0$  and  $b = n$ . Using

knot Floer homology diagrams, we shall show the following identities:

$$(6.1) \quad 0 = d(S_{-2n}^3(-2D_k), \mathfrak{s}_b) - d(S^3, \mathfrak{s}_0) - \epsilon_1$$

$$(6.2) \quad 0 = d(S_{-2n}^3(-U), \mathfrak{s}_b) - d(S^3, \mathfrak{s}_0) - \epsilon_1$$

$$(6.3) \quad -2k = d(S_{-2n, 2n}^3(-2D_k, 2D_k), \mathfrak{s}_n) - d(S_{-2n}^3(-2D_k), \mathfrak{s}_b) - \epsilon_2$$

$$(6.4) \quad 0 = d(S_{-2n, 2n}^3(-U, U), \mathfrak{s}_n) - d(S_{-2n}^3(-U), \mathfrak{s}_b) - \epsilon_2,$$

where  $\mathfrak{s}_0$  is the unique spin structure of  $S^3$  and  $\epsilon_i$  are grading shifts. The grading shifts  $\epsilon_i$  are homological invariants [28] and hence (6.1) and (6.3), respectively, have the same grading shifts  $\epsilon_1$  and  $\epsilon_2$  as (6.2) and (6.4), respectively.

Ozsváth and Szabó [30, Corollary 4.2] showed that, for a knot  $K$  in  $S^3$ , the complex  $CF^+(S_{-2n}^3(K), \mathfrak{s}_b)$  is filtered chain homotopic to

$$CFK^\infty(S^3, K)/CFK^\infty(S^3, K)_{\{i < 0 \cup j < -b\}}[\epsilon],$$

where the grading shift  $\epsilon$  is  $d(S^3, \mathfrak{s}_0) + \epsilon_1$  according to [28]. We remark that, though  $d(S^3, \mathfrak{s}_0) = 0$ , it is written explicitly in order to indicate the  $d$ -invariant of the base space where the knot lies. Proposition 6.1 allows us to replace  $-2D_k$  with  $-T_{2, 4k+1}$ . We see that in the complex  $CFK^\infty(S^3, -T_{2, 4k+1})$  the cycle at filtration level  $(0, -2k)$  is the cycle of grading zero having the least  $j$ -filtration among all grading zero cycles, and all cycles of grading less than zero have  $i$ -filtration less than zero. Since  $-2k > -n = -b$ , the cycle at filtration level  $(0, -2k)$  lives and all the cycles of grading less than zero vanish in the quotient. See Figure 11 for the case  $n = 4$  and  $k = 1$ .

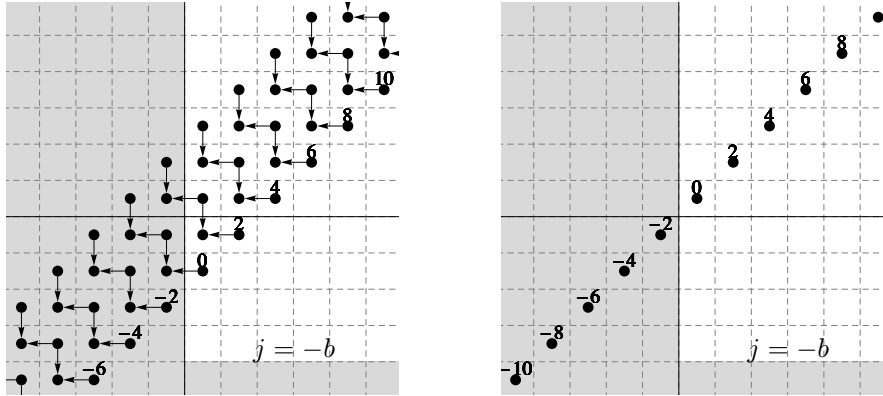


FIGURE 11.

This shows the identity  $d(S_{-2n}^3(-2D_k), \mathfrak{s}_b) - d(S^3, \mathfrak{s}_0) - \epsilon_1 = 0$ . A similar argument shows  $d(S_{-2n}^3(-U), \mathfrak{s}_b) - d(S^3, \mathfrak{s}_0) - \epsilon_1 = 0$ . These two identities give rise to:

**Proposition 6.5.**  $d(S_{-2n}^3(-2D_k), \mathfrak{s}_b) - d(S_{-2n}^3(-U), \mathfrak{s}_b) = 0$ .

By Theorem 6.3, noting  $b = n > 2k$ , we can identify

$$CFK^\infty(S_{-2n}^3(-2D_k), 2D_k, \mathfrak{s}_b)_{i,j} = CFK^\infty(S^3, T_{2, 4k+1})_{i,j+1} \oplus A,$$

where  $A$  is an acyclic complex. That is, the complex  $CFK^\infty(S^3_{-2n}(-2D_k), 2D_k, \mathfrak{s}_b)$  is filtered chain homotopic to  $CFK^\infty(S^3, T_{2,4k+1})$  with  $j$ -filtration shifted downward by 1 plus an acyclic complex  $A$ . Combining this with Theorem 4.1, we have that  $CF^+(S^3_{-2n,2n}(-2D_k, 2D_k), \mathfrak{s}_n)$  is filtered chain homotopic to

$$CFK^\infty(S^3, T_{2,4k+1})_{i,j+1} / CFK^\infty(S^3, T_{2,4k+1})_{i,j+1\{i<0, j<a\}}[\epsilon],$$

where  $a = 0$  and  $\epsilon = d(S^3_{-2n}(-2D_k), \mathfrak{s}_b) + \epsilon_2$  for some  $\epsilon_2$  independent of  $T_{2,4k+1}$ . The cycles  $x$  at filtration level  $(i, 2k - i - 1)$ ,  $0 \leq i \leq 2k$ , are all of the grading zero cycles in  $CFK^\infty(S^3, T_{2,4k+1})_{i,j+1}$ . It is easy to see that the cycles  $U^k x$  at filtration level  $(i', -i' - 1)$ ,  $-k \leq i' \leq -k$ , have grading  $-2k$  and none of them vanish in the quotient, while at least one of  $U^{k'} x$  vanishes in the quotient if  $k' > k$ . See Figure 12 for the case  $n = 4$  and  $k = 1$ .

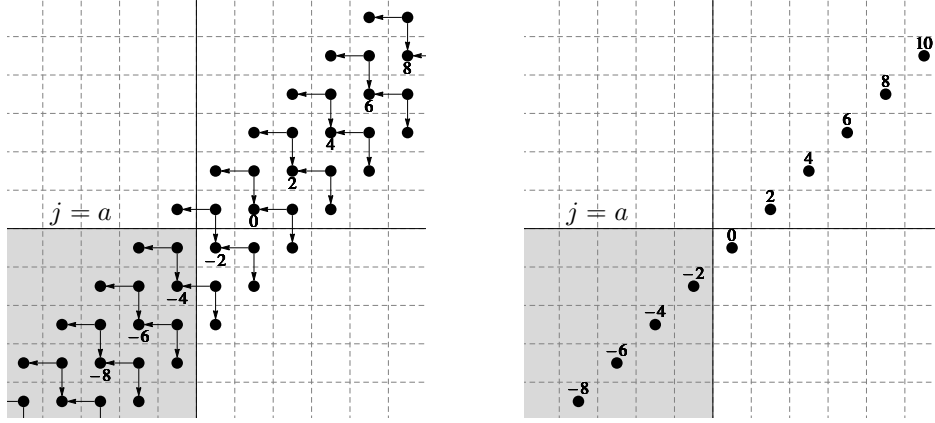


FIGURE 12.

This implies the identity

$$d(S^3_{-2n,2n}(-2D_k, 2D_k), \mathfrak{s}_n) - d(S^3_{-2n}(-2D_k), \mathfrak{s}_b) - \epsilon_2 = -2k.$$

Combining Theorems 6.4 and 4.1, a similar argument as done above tells us that

$$d(S^3_{-2n,2n}(-U, U), \mathfrak{s}_n) - d(S^3_{-2n}(-U), \mathfrak{s}_b) - \epsilon_2 = 0.$$

Taking difference of the above two identities and applying Proposition 6.5 we have:

**Proposition 6.6.**  $d(M(K_{D_k,n}), \mathfrak{s}_n) - d(M(K_{U,n}), \mathfrak{s}_n) = -2k$ .

Now we can determine  $k_n$ .

**Theorem 6.7.** *For any  $n \in \mathcal{N}$ , there is a positive integer  $k_n$  and a  $\text{Spin}^c$  structure  $\mathfrak{s}_x$  such that  $d(M(K_{D_{k_n},n}), \mathfrak{s}_x) \neq d(M(K_{U,n}), \mathfrak{s}_{b'x})$  for any  $b'$  such that  $b'^2 \equiv 1 \pmod{4n^2 + 1}$ .*

**Proof.** Let  $n \in \mathcal{N}$  and let  $x = c_1(\mathfrak{s}_n)$ , where  $\mathfrak{s}_n$  is the  $\text{Spin}^c$  structure in Proposition 6.6. As  $k$  varies over  $0 \leq k < n/2$ , there are  $\lfloor n/2 \rfloor$  number of possible values of  $d(M(K_{D_k,n}), \mathfrak{s}_x)$ , while  $d(M(K_{U,n}), \mathfrak{s}_{b'x})$  has at most  $\lfloor n/2 \rfloor - 1$  number of possible values as  $b'$  varies as observed in Subsection 5.4. Therefore, we can find  $k_n$  for which the  $d(M(K_{D_{k_n},n}), \mathfrak{s}_x)$  does not match with  $d(M(K_{U,n}), \mathfrak{s}_{b'x})$  for any  $b'$  such that  $b'^2 \equiv 1 \pmod{4n^2 + 1}$ .  $\square$

APPENDIX A. THE HOMOLOGY  $CFK^\infty(S^3, Wh(T_{2,3}, 0))$ 

In this section of the appendix we prove:

**Proposition 6.1.** *The chain complex  $CFK^\infty(S^3, D)$  is chain homotopy equivalent to the chain complex  $CFK^\infty(S^3, T_{2,3}) \oplus A$ , where  $A$  is an acyclic complex. The presence of the acyclic summand does not change the relevant width:*

$$w(CFK^\infty(S^3, D)) = w(CFK^\infty(S^3, T_{2,3}) \oplus A).$$

In order to prove this proposition, we need the following well-known lemma about how a basis change affects the 2-dimensional diagram of a knot Floer complex. See [12, Lemma 6.1] for instance.

**Lemma A.1.** *Let  $C_*$  be a knot Floer complex with a 2-dimensional arrow diagram  $D$  given by an  $\mathbb{F}$ -basis. Suppose that  $x, y$  are two basis elements of the same grading such that each of the  $i$  and  $j$  filtrations of  $x$  is not greater than that of  $y$ . Then the basis change given by  $y' = y + x$  gives rise to a diagram  $D'$  of  $C_*$  which differs from  $D$  only at  $y$  and  $x$  as follows:*

- Every arrow from some  $z$  to  $y$  in  $D$  adds an arrow from  $z$  to  $x$  in  $D'$ .
- Every arrow from  $x$  to some  $w$  in  $D$  adds an arrow from  $y'$  to  $w$  in  $D'$ .

**Proof.** First note that this basis change does not alter the grading or double filtrations. If  $\partial z = y + A$  for  $z, A \in C_*$ , then  $\partial z = y' + x + A$ , which shows that every arrow from  $z$  to  $y$  should add an arrow from  $z$  to  $x$ . Since  $\partial y' = \partial y + \partial x$ , every arrow from  $x$  should add an arrow from  $y'$ . See Figure 13 for an example.  $\square$

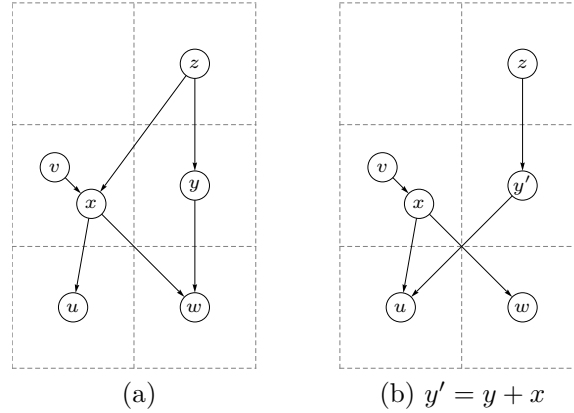


FIGURE 13.

*Proof of Proposition 6.1.* Theorem 1.2 of [12] shows that

$$\widehat{HFK}_*(D, j) = \begin{cases} \mathbb{F}_{(-1)}^2 \oplus \mathbb{F}_{(0)}^2, & j = 1 \\ \mathbb{F}_{(-2)}^4 \oplus \mathbb{F}_{(-1)}^3, & j = 0 \\ \mathbb{F}_{(-3)}^2 \oplus \mathbb{F}_{(-2)}^2, & j = -1 \\ 0, & \text{otherwise.} \end{cases}$$

We assign  $\mathbb{F}$ -bases to each summand in the direct sum decomposition as follows:

$$\widehat{HFK}_*(D, j) = \begin{cases} \langle u_1, u_2 \rangle \oplus \langle x_1, x_2 \rangle, & j = 1 \\ \langle v_1, v_2, v_3, v_4 \rangle \oplus \langle y_1, y_2, y_3 \rangle, & j = 0 \\ \langle w_1, w_2 \rangle \oplus \langle z_1, z_2 \rangle, & j = -1. \end{cases}$$

Following Rasmussen [37, Lemma 4.5] (or [12, Lemma 5.3]),  $\widehat{HFK}_*(D)$  is chain homotopy equivalent to the  $\widehat{CFK}(D)$ . So we assume that  $CFK^\infty(D)_{0,j} = \widehat{HFK}_*(D, j)$  and  $CFK^\infty(D)_{i,j} \cong U^{-i} CFK^\infty(D)_{0,j} = \widehat{HFK}_*(D, j-i)$ . If necessary, we put the grading in the superscript of the generator; for instance,  $x_1^2$  denotes the grading 2 generator among  $U^i x_1$  for  $i \in \mathbb{Z}$ .

First note that there are no components of boundary maps between generators of the same  $(i, j)$ -filtration since they would be reduced in  $\widehat{HFK}_*(D, j)$ . If we denote the vertical, horizontal, and diagonal components of the boundary map  $\partial$  of  $CFK^\infty(D)$  by  $\partial_V$ ,  $\partial_H$ , and  $\partial_D$ , respectively, then  $\partial = \partial_V + \partial_H + \partial_D$ . We will determine  $\partial$  in the order of  $\partial_V$ ,  $\partial_H$  and  $\partial_D$ .

Note that  $\mathbb{F}_{(0)}^2 \xrightarrow{\partial_V} \mathbb{F}_{(-1)}^3 \xrightarrow{\partial_V} \mathbb{F}_{(-2)}^2$ , or,  $\langle x_1, x_2 \rangle \xrightarrow{\partial_V} \langle y_1, y_2, y_3 \rangle \xrightarrow{\partial_V} \langle z_1, z_2 \rangle$  is a chain subcomplex of  $\widehat{CFK}(D)$  since  $\partial$  lowers the grading by one. Since  $\widehat{HF}(S^3) = \mathbb{F}_{(0)}$  is generated by a grading 0 cycle, we have that the subcomplex is right exact. Changing bases, we may assume that  $\partial_V(x_1) = \partial_V(y_1) = \partial_V(z_1) = \partial_V(z_2) = 0$ ,  $\partial_V(x_2) = y_1$ ,  $\partial_V(y_1) = 0$ ,  $\partial_V(y_2) = z_1$ , and  $\partial_V(y_3) = z_2$ . See Figure 14(b).

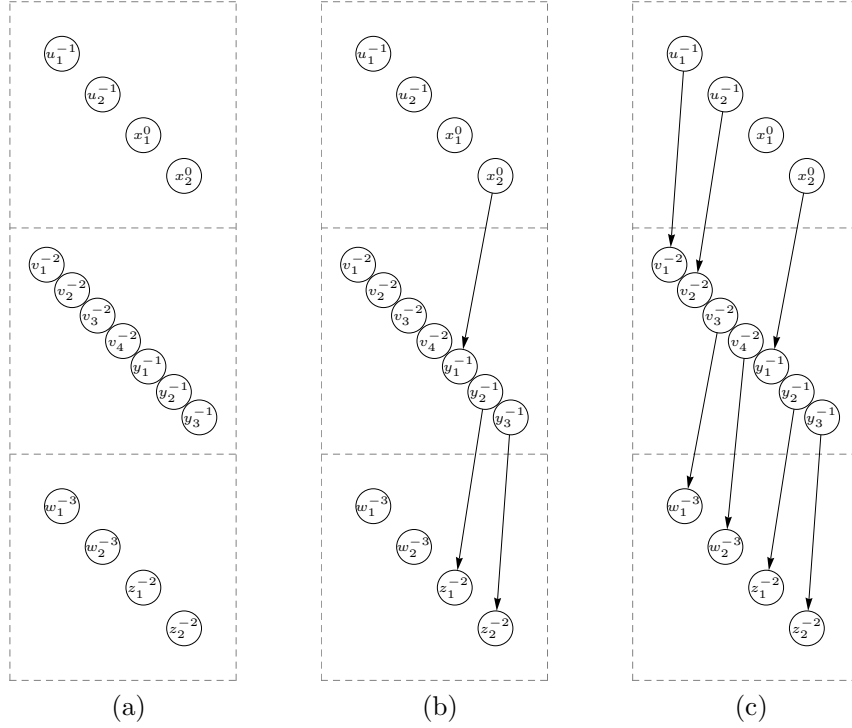


FIGURE 14.



We will find  $\partial_V(u_1)$ , which should lie in  $\langle v_1, v_2, v_3, v_4, z_1, z_2 \rangle$ . If  $\partial_V(u_1) = az_1 + bz_2 \in \langle z_1, z_2 \rangle$  for  $a, b \in \mathbb{F}$ , then  $u_1 + ay_1 + by_2$  represents a nontrivial element of grading  $-1$  in  $\widehat{HF}(S^3)$ , which is impossible. Thus  $\partial_V(u_1)$  must have a nontrivial component in  $\langle v_1, v_2, v_3, v_4 \rangle$ , which may be assumed to be  $v_1$  by changing basis for  $\langle v_1, v_2, v_3, v_4 \rangle$ . If  $\partial_V(u_1) = v_1 + az_1 + bz_2$ , then the change of basis  $v'_1 = v_1 + az_1 + bz_2$  gives rise to  $\partial_V(u_1) = v'_1$  and  $\partial_V v'_1 = \partial_V v_1$  as in Lemma A.1. So we may assume that  $\partial_V u_1 = v_1$  and  $\partial_V u_2 = v_2$  similarly. The image of  $\langle v_3, v_4 \rangle$  under  $\partial_V$  should be equal to  $\langle w_1, w_2 \rangle$  since  $\widehat{HF}(S^3) = \mathbb{Z}$  in which  $v_3, v_4, w_1$  and  $w_2$  should vanish. So  $\{\partial_V(v_3), \partial_V(v_4)\}$  is a basis for  $\langle w_1, w_2 \rangle$  and we may assume that  $w_1 = \partial_V(v_3)$  and  $w_2 = \partial_V(v_4)$ . The vertical components of the boundary maps are all determined as shown in Figure 14(c).

Next, we will determine the horizontal components of the boundary map of  $CFK^\infty(D)$  of which each column looks like Figure 14. We will argue that the complex will have a 2-dimensional illustration described in Figure 15. Analogously as in the vertical case, note that  $\langle z_1, z_2 \rangle \xrightarrow{\partial_H} \langle y_1, y_2, y_3 \rangle \xrightarrow{\partial_H} \langle x_1, x_2 \rangle$  is a chain subcomplex  $S$  of  $CFK^\infty(D)_{\{j \leq 0\}} / CFK^\infty(D)_{\{j < 0\}}$  since  $\partial$  lowers the degree by one. Observe as well that for any  $s \in S$ , one less graded elements of  $s$  are either to the left or below and hence  $\partial s = \partial_V s + \partial_H s$ . In particular there are no diagonal components of the boundary maps restricted to  $S$ . This implies that  $\partial x_1 = \partial_V x_1 = 0$  and  $\partial x_2 = \partial_V x_2 = y_1$ .

Since  $\widehat{HF}(S^3) \cong \mathbb{F}_{(0)}$  is isomorphic to  $H_*(CFK^\infty(D)_{\{j \leq 0\}} / CFK^\infty(D)_{\{j < 0\}})$ , we have that the subcomplex is right exact. We may choose a  $\mathbb{F}$ -basis  $\{z_1, z_2\}$  so that  $\partial_H(z_1) = 0$ . To keep the same vertical description as in Figure 14(c), we adjust the basis for  $\langle y_2, y_3 \rangle$  accordingly. Observe that  $\partial z_2$  has no diagonal arrows since one less graded elements are located only to the left. So we have  $\partial z_2 \in \langle y_1, y_2, y_3 \rangle$ . If  $\partial z_2$  is of the form  $y_2 + B$  for  $B \in \langle y_1, y_3 \rangle$ , then  $0 = \partial^2 z_2 = \partial y_2 + \partial B = z_1 + \partial_H y_2 + \partial B$ , which, on the other hand, can never be zero since  $\partial_H y_2 \in \langle x_1, x_2 \rangle$ ,  $\partial B \in \langle \partial y_1, \partial y_3 \rangle = \langle z_2 \rangle$ , and  $z_1$  does not belong to  $\langle x_1, x_2, z_2 \rangle$ . Thus  $\partial z_2$  does not have  $y_2$ . Note that it does not have  $y_3$  similarly. Thus  $\partial z_2$  must be  $y_1$ .

This also implies that  $\partial_H \langle y_2, y_3 \rangle = \langle x_1, x_2 \rangle$ . If  $\partial_H y_2$  is of the form  $x_2 + ax_1$  for  $a \in \mathbb{F}$ , then  $\partial y_2 = (\partial_H + \partial_V)y_2 = x_2 + ax_1 + z_1$  and  $0 = \partial^2 y_2 = \partial(x_2 + ax_1 + z_1) = y_1 \neq 0$ , which is impossible. Thus we have  $\partial_H y_2 = x_1$ . Then  $\partial_H y_3$  should be of the form  $x_2 + ax_1$ . By changing basis  $x'_2 = x_2 + ax_1$  we may assume  $\partial_H y_3 = x_2$ . We have a diagram for  $CFK^\infty(D)$  as in Figure 15 with only vertical and horizontal components of the boundary maps.

Finally, we will deal with the diagonal components of the boundary maps. As mentioned earlier, due to grading constraints there are no diagonal maps coming from generators  $x$ 's,  $y$ 's and  $z$ 's while there may be diagonals going in. On the other hand, there are no diagonal maps going into generators  $u$ 's,  $v$ 's and  $w$ 's. All possible cases of diagonal maps are: (1) from  $u$ 's to  $x$ 's, (2) from  $v$ 's to  $y$ 's, and (3) from  $w$ 's to  $z$ 's. This implies that the complex  $T$  generated by  $x_1, y_2$  and  $z_1$  is indeed a subcomplex of  $CFK^\infty(D)$ .

We will show that proper basis changes can eliminate all the diagonal arrows going into  $T$ . Then  $CFK^\infty(D)$  splits into  $\mathbb{F}[U, U^{-1}]T$  and a subcomplex  $A$ . Note that  $\mathbb{F}[U, U^{-1}]T$  is isomorphic to  $CFK^\infty(T(2, 3))$  and  $A$  is acyclic, i.e.,  $H_*(A) = 0$ ; For, [29, Section 10] showed  $HF^\infty(S^3) \cong \mathbb{F}[U, U^{-1}]$  and  $HF^\infty(S^3) \cong H_*(CFK^\infty(D)) \cong H_*(\mathbb{F}[U, U^{-1}]T)$  as  $\mathbb{F}[U, U^{-1}]$ -modules.

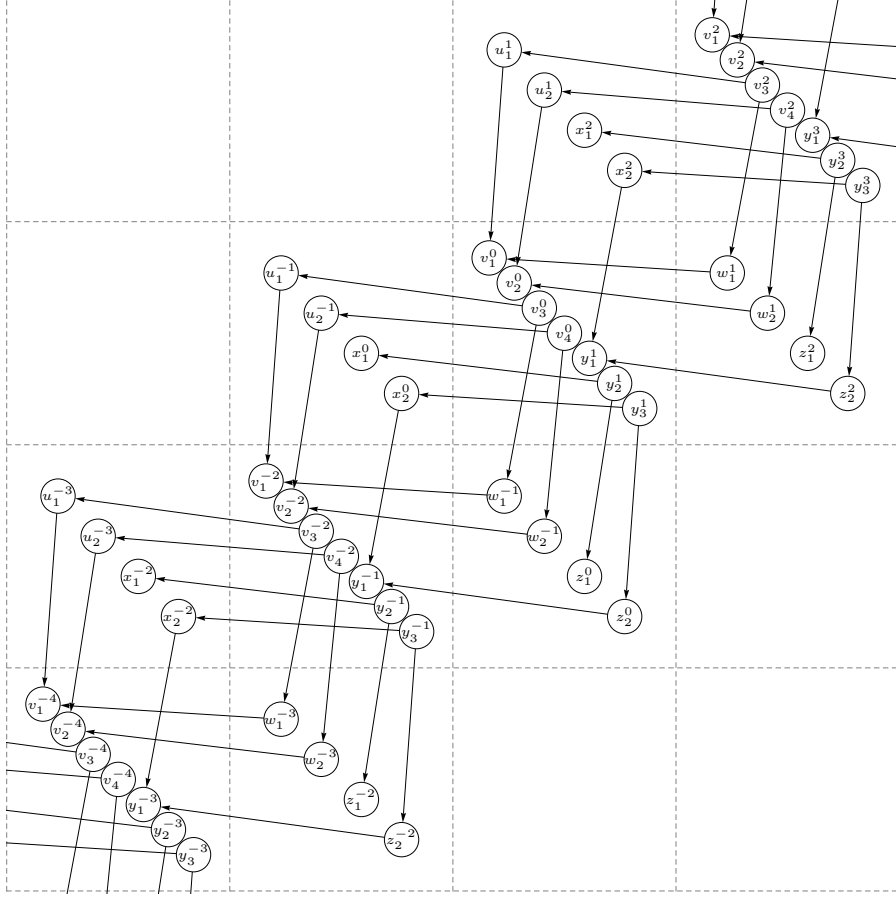


FIGURE 15.

First, we show that  $\partial v_1$  and  $\partial v_2$  cannot include  $y_2$ . Note that they have zero vertical and horizontal components. If  $\partial v_1 = y_2 + ay_1 + by_3$  for  $a, b \in \mathbb{F}$ , then  $0 = \partial^2 v_1 = x_1 + z_1 + bx_2 + bz_2$  which cannot be zero for any  $a, b$ . So there are no arrows from  $v_1$  or  $v_2$  to  $y_2$ .

We claim that, for any  $a, b \in \mathbb{F}$  and  $i = 1, 2$ , the following are equivalent:

- (1)  $\partial_D u_i = ax_1 + bx_2$
- (2)  $\partial_D v_{i+2} = ay_2 + by_3 + cy_1$  for some  $c \in \mathbb{F}$
- (3)  $\partial_D w_i = az_1 + bz_2$ .

We prove the claim only for  $i = 1$ ; almost the same argument applies to  $i = 2$ . Let  $\partial_D u_1 = ax_1 + bx_2$ ,  $\partial_D v_3 = c_1 y_1 + c_2 y_2 + c_3 y_3$ , and  $\partial_D w_1 = d_1 z_1 + d_2 z_2$  for  $a, b, c_*, d_* \in \mathbb{F}$ . The constraint  $\partial^2 = 0$  gives rise to the equalities

$$\begin{aligned}
 0 &= \partial^2 v_3 = \partial(u_1 + w_1 + c_1 y_1 + c_2 y_2 + c_3 y_3) \\
 &= (v_1 + ax_1 + bx_2) + (v_1 + d_1 z_1 + d_2 z_2) + c_2(x_1 + z_1) + c_3(x_2 + z_2) \\
 &= (a + c_2)x_1 + (b + c_3)x_2 + (d_1 + c_2)z_1 + (d_2 + c_3)z_2.
 \end{aligned}$$

Thus  $a = c_2 = d_1$  and  $b = c_3 = d_2$ .

Suppose  $a = 1$  for  $i = 1$ . Let  $v'_1 = v_1 + x_1$  and  $w'_1 = w_1 + y_2$ . Then all arrows going into  $v_1$  or  $w_1$  come from  $u_1$ ,  $w_1$  or  $v_3$  and hence we need to check the boundaries of  $u_1$ ,  $v_3$ ,  $w'_1$  and  $v'_1$ .  $\partial u_1 = v'_1 + bx_2$ ,  $\partial v_3 = u_1 + w'_1 + by_3 + cy_1$ ,  $\partial w'_1 = \partial w_1 + \partial y_2 = (v_1 + z_1 + bz_2) + (x_1 + z_1) = v_1 + bz_2$ , and  $\partial v'_1 = \partial v_1 + \partial x_1 = 0$ . With these new basis elements  $v'_1$  and  $w'_1$  there are no diagonal components from  $u_1, v_3, w_1$  to  $x_1, y_2, z_1$ . Similar argument works for  $u_2, v_4, w_2$ . Thus  $T$  can be assumed to be a direct summand as desired.  $\square$

We remark that a similar process of changing bases in the previous proof permits us to assume that  $CFK^\infty(S^3, D)$  looks like Figure 15. But it is unnecessary to present the proof in order to complete our task for this manuscript.

#### APPENDIX B. $CFK^\infty(S^3, T_{2,2k+1})$

**Theorem B.1.**  $CFK^\infty(S^3, T_{2,3})^{\otimes k} = CFK^\infty(S^3, T_{2,2k+1}) \oplus A$  where  $A$  is acyclic. The presence of the acyclic summand does not change the relevant width:

$$w(CFK^\infty(S^3, T_{2,2k+1})) = w(CFK^\infty(S^3, T_{2,2k+1}) \oplus A).$$

**Proof.** The proof is by induction. We show that

$$CFK^\infty(S^3, T_{2,2k+1}) \otimes CFK^\infty(S^3, T_{2,3}) = CFK^\infty(S^3, T_{2,2k+3}) \oplus A.$$

The complex  $CFK^\infty(S^3, T_{2,2k+1})$  has filtered generators at grading 0:  $[x, i, j]$  where  $i \geq 0$ ,  $j \geq 0$  and  $i + j = k$ . There are also generators at grading level 1,  $[y, i, j]$  with  $i \geq 1, j \geq 1$  and  $i + j = k + 1$ . The boundary map is given by  $\partial[y, i, j] = [x, i - 1, j] + [x, i, j - 1]$ . (Notice that the symbols  $x$  and  $y$  do not correspond to intersection points in a Heegaard diagram. The  $i$  and  $j$  denote the filtration levels.)

In order to distinguish the complex for  $T_{2,3}$ , we replace  $x$  and  $y$  with  $z$  and  $w$ , so that the complex is generated by  $[z, 0, 1]$ ,  $[z, 1, 0]$ , and  $[w, 1, 1]$ .

The tensor product  $CFK^\infty(S^3, T_{2,2k+1}) \otimes CFK^\infty(S^3, T_{2,3})$  has generators of type  $x \otimes z$  at grading level 0,  $x \otimes w$  and  $y \otimes z$  at grading level 1, and  $y \otimes w$  at grading level 2. In total there are  $3(2k + 1)$  generators.

We now make a basis change, replacing certain generators with their sums with other generators, relabeled as indicated:

- $[x, i, j] \otimes [w, 1, 1] \rightarrow [x, i, j] \otimes [w, 1, 1] + [y, i + 1, j] \otimes [z, 0, 1] = \alpha_i$ , for all  $0 \leq i < k$ .
- $[x, i, j] \otimes [z, 1, 0] \rightarrow [x, i, j] \otimes [z, 1, 0] + [x, i + 1, j - 1] \otimes [z, 0, 1] = \beta_i$ , for all  $0 \leq i < k$ .
- $[y, i, j] \otimes [z, 1, 0] \rightarrow [y, i, j] \otimes [z, 1, 0] + [x, i, j - 1] \otimes [w, 1, 1] = \gamma_i$ , for all  $0 < i \leq k$ .

Now we isolate out acyclic pieces, using the following four observations.

- $\partial[y, i, j] \otimes [w, 1, 1] = [x, i - 1, j] \otimes [w, 1, 1] + [x, i, j - 1] \otimes [w, 1, 1] + [y, i, j] \otimes [z, 0, 1] + [y, i, j] \otimes [z, 1, 0] = \alpha_{i-1} + \gamma_i$ .
- $\partial\alpha_{i-1} = \partial([x, i - 1, j] \otimes [w, 1, 1] + [y, i, j] \otimes [z, 0, 1]) = [x, i - 1, j] \otimes [z, 0, 1] + [x, i - 1, j] \otimes [z, 1, 0] + [x, i - 1, j] \otimes [z, 0, 1] + [x, i, j - 1] \otimes [z, 0, 1] = [x, i - 1, j] \otimes [z, 1, 0] + [x, i, j - 1] \otimes [z, 0, 1] = \beta_{i-1}$ .

- $\partial\gamma_i = [x, i-1, j] \otimes [z, 1, 0] + [x, i, j-1] \otimes [z, 1, 0] + [x, i, j-1] \otimes [z, 0, 1] + [x, i, j-1] \otimes [z, 1, 0] = [x, i-1, j] \otimes [z, 1, 0] + [x, i, j-1] \otimes [z, 0, 1] = \beta_{i-1}$ .
- $\partial\beta_{i-1} = 0$ .

From this we see that there is an acyclic *summand*

$$\langle [y, i, j] \otimes [w, 1, 1] \rangle \xrightarrow{\partial} \langle \alpha_{i-1}, \gamma_i \rangle \xrightarrow{\partial} \langle \beta_i \rangle.$$

For instance, see Figure 16 for the case  $k = 2$ .

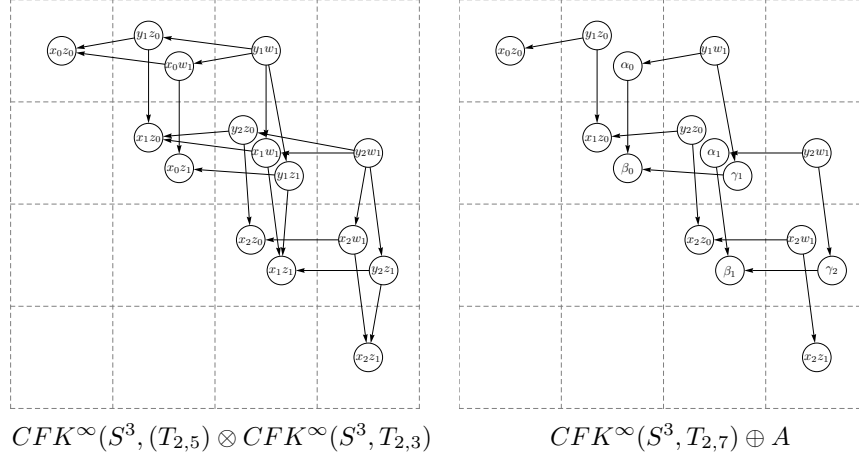


FIGURE 16. Notation:  $x_i z_{i'} = [x, i, k-i] \otimes [z, i', 1-i']$ ,  $x_i w_1 = [x, i, k-i] \otimes [w, 1, 1]$ ,  $y_i z_{i'} = [y, i, k+1-i] \otimes [z, i', 1-i']$ ,  $y_i w_1 = [y, i, k+1-i] \otimes [w, 1, 1]$ ,  $\alpha_i = x_i w_1 + y_{i+1} z_0$ ,  $\beta_i = x_i z_1 + x_{i+1} z_0$ , and  $\gamma_i = y_i z_1 + x_i w_1$ .

There are  $k$  such summands, with a total rank of  $4k$ . The original complex had rank  $3(2k+1) = 6k+3$ . Thus, splitting off the acyclic summands leaves a complex of rank  $2k+3$ . Generators for a complement to the acyclic summand are given by the set  $\{[x, i, j] \otimes [z, 0, 1], [y, i, j] \otimes [z, 0, 1]\}$  and two more elements,  $[x, k, 0] \otimes [w, 1, 1]$  and  $[x, k, 0] \otimes [z, 1, 0]$ . Finally, we note that this is a subcomplex of the desired isomorphism type, as follows from three simple observations:  $\partial([x, i, j] \otimes [z, 0, 1]) = 0$ ,  $\partial([y, i, j] \otimes [z, 0, 1]) = [x, i-1, j] \otimes [z, 0, 1] + [x, i, j-1] \otimes [z, 0, 1]$  and  $\partial([x, k, 0] \otimes [w, 1, 1]) = [x, k, 0] \otimes [z, 0, 1] + [x, k, 0] \otimes [z, 1, 0]$ .  $\square$

Note that similar computation have recently appeared in [16].

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